Integración por sustitución o cambio de variable

a) $$\int \sqrt[3]{x+5}\,dx = \int (x+5)^{\frac{1}{3}}dx = \frac{(x+5)^{\frac{4}{3}}}{\frac{4}{3}} + C$$

b) $$\int \sqrt[5]{3+\boxed{4}x}\,dx = \int (3+4x)^{\frac{1}{5}}\,dx = \boxed{\frac{1}{4}}\,\frac{(3+4x)^{\frac{6}{5}}}{\frac{6}{5}} + C$$

c) $$\int \sec^2 \boxed{5}x\,dx = \boxed{\frac{1}{5}}\,tg\,5x + C$$

d) $$\int \cos \boxed{2}x\,dx = \boxed{\frac{1}{2}}\,\text{sen}\,2x + C$$

e) $$\int \frac{sen\,2x}{1+\cos^2 x}\,dx = -\int \frac{\cancel{\text{sen}\,2x}}{t}\,\frac{dt}{\cancel{\text{sen}\,2x}} = -\int \frac{dt}{t} = -\ln t = -\ln\left(1+\cos^2 x\right) + C$$

$$\begin{cases} t = 1+\cos^2 x \\ dt = -2\cos x\,\text{sen}\,x\,dx \end{cases}$$

f) $$\int \frac{e^{2x}}{1+e^x}\,dx = \int \frac{(t-1)^2}{t}\,\frac{dt}{\cancel{t-1}} = \int \frac{t-1}{t}\,dt = \int dt - \int \frac{1}{t}\,dt = t - \ln t = 1+e^x - \ln\left(1+e^x\right) + C$$

$$\begin{cases} t = 1+e^x; \\ dt = e^x dx; \qquad dx = \dfrac{dt}{t-1} \end{cases}$$

g) $$\int \frac{2\cos x}{4+\text{sen}\,x}\,dx = 2\int \frac{\cancel{\cos t}}{t}\,\frac{dt}{\cancel{\cos t}} = 2\int \frac{dt}{t} = 2\ln t = 2\ln\left(4+\text{sen}\,x\right) + C$$

$$\begin{cases} t = 4+\text{sen}\,x \\ dt = \cos x\,dx \end{cases}$$

h) $$\int \text{sen}^2 \tfrac{x}{2}\cos\tfrac{x}{2}\,dx = 2\int t^2\,\cancel{\cos\tfrac{x}{2}}\,\frac{dt}{\cancel{\cos\tfrac{x}{2}}} = \frac{2t^3}{3} = \frac{2}{3}\,\text{sen}^3\tfrac{x}{2} + C$$

$$\begin{cases} t = \text{sen}\,\tfrac{x}{2} \\ dt = \tfrac{1}{2}\cos\tfrac{x}{2}\,dx;\; dx = \dfrac{2dt}{\cos\tfrac{x}{2}} \end{cases}$$

i) $$\int \left(1+\text{tg}^2 x\right)dx = \int \left(1+\text{tg}^2 x + 2\text{tg}x\right)dx = \int \left(\sec^2 x + 2\,\frac{\text{sen}\,x}{\cos x}\right)dx = \text{tg}x + 2\ln x + C$$

j) $$\int \frac{\sec^2 x}{a+b\,\text{tg}\,x}\,dx = \frac{1}{b}\int \frac{b\sec^2 x}{a+b\,\text{tg}\,x}\,dx = \frac{1}{b}\ln\left(a+b\,\text{tg}\,x\right) + C$$

k) $$\int \frac{\text{sen}\,3x}{\sqrt{5+\cos 3x}}\,dx = -\frac{1}{3}\int \frac{\cancel{\text{sen}\,3x}}{t^{\frac{1}{2}}}\,\frac{dt}{\cancel{\text{sen}\,3x}} = -\frac{1}{3}\,\frac{t^{1/2}}{\frac{1}{2}} = -\frac{2}{3}\sqrt{5+\cos x} + C$$

$$\begin{cases} 5+\cos 3x = t; \\ -3\,\text{sen}\,3x\,dx = dt; \quad dx = \dfrac{dt}{-3\,\text{sen}\,3x} \end{cases}$$

l) $$\int \frac{x^2}{\left(3+2x^3\right)^2}dx = \frac{1}{6}\int \frac{\cancel{x^2}}{t^2}\frac{dt}{\cancel{x^2}} = \frac{1}{6}\int t^{-2}dt = -\frac{1}{6t} = -\frac{1}{6\left(3+2x^3\right)^2} + C$$

$$\begin{cases} 3+2x^3 = t; \\ 6x^2dx = dt; \quad dx = \dfrac{dt}{6x^2} \end{cases}$$

m) $$\int \frac{ax^m}{1+b\,x^{n+1}}dx = \frac{a}{(n+1)b}\int \frac{\cancel{x^n}}{t}\frac{dt}{\cancel{x^n}} = \frac{a}{(n+1)b}\int \frac{dt}{t} =$$
$$= \frac{a}{(n+1)b}\ln t = \frac{a}{(n+1)b}\ln\left(1+bx^{n+1}\right) + C$$

$$\begin{cases} 1+b\,x^{n+1} = t; \\ (n+1)bx^n dx = dt; \quad dx = \dfrac{dt}{(n+1)bx^n} \end{cases}$$

n) $$\int \frac{e^x}{4-3e^x}dx = \frac{1}{3}\int \frac{\cancel{e^x}}{t}\frac{dt}{\cancel{e^x}} = \frac{1}{3}\ln t = \frac{1}{3}\ln\left(4-3e^x\right) + C$$

$$\begin{cases} 4-3e^x; \\ -3e^x dx = dt; \quad dx = \dfrac{dx}{-3e^x} \end{cases}$$

Integración por partes

Sean las funciones $u = f(x)$ y $v = g(x)$; $d(u\cdot v) = u\ dv + v\ du \Rightarrow$ u $dv = d(u\cdot v) - v\ du \Rightarrow$
$\Rightarrow \int u\,dv = \int d(uv) - \int v\,du$

Se hacen dos partes del integrando, una de ellas se iguala a u y la otra junto con dx a dv, la parte igualada a dv debe ser fácilmente integrable y la nueva integral que aparece tiene que ser igual o menos complicada que la dada al principio.

a) $$\int x^2\cos x\,dx = x^2\text{sen}\,x - 2\int x\,\text{sen}\,x\,dx = x^2\text{sen}\,x - 2\left(-x\cos x - \int \cos x dx\right) =$$
$$= x^2\text{sen}\,x + 2x\cos x - 2\,\text{sen}\,x + C$$

$u = x^2$	$du = 2xdx$
$dv = \cos xdx$	$v = \int \cos xdx = \text{sen}\,x$

$u = x$	$du = dx$
$dv = \text{sen}\,xdx$	$v = \int \text{sen}\,xdx = -\cos x$

b) $$\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C$$

$u = \ln x$	$du = \dfrac{1}{x}dx$
$dv = dx$	$v = \int dx = x$

c) $\int x\ln x\,dx=\frac{x^2}{2}\ln x-\frac{1}{2}\int x\,dx=\frac{x^2}{2}\ln x-\frac{x^2}{4}+C$

$u=\ln x$	$du=\frac{1}{x}dx$
$dv=x\,dx$	$v=\int x\,dx=\frac{x^2}{2}$

d) $\int x^2e^{2x}dx=\frac{x^2}{2}e^{2x}-\int xe^{2x}dx=\frac{x^2}{2}e^{2x}-\left(\frac{x}{2}e^{2x}-\frac{1}{2}\int e^{2x}dx\right)=$

$=\frac{x^2}{2}e^{2x}-\frac{x}{2}e^{2x}+\frac{1}{4}e^{2x}=e^{2x}\left(\frac{x^2}{2}-\frac{x}{2}+\frac{1}{4}\right)+C$

$u=x^2$	$du=2x\,dx$
$dv=e^{2x}dx$	$v=\frac{1}{2}e^{2x}$

$u=x$	$du=dx$
$dv=e^{2x}dx$	$v=\frac{1}{2}e^{2x}$

e) $\int x\sqrt{1+x}dx=2\int\left(t^2-1\right)\cdot t\cdot t\,dt=2\int\left(t^4-t^2\right)dt=$

$=2\left(\frac{t^5}{5}-\frac{t^3}{3}\right)=\frac{2\left(\sqrt{1+x}\right)^5}{5}-\frac{2\left(\sqrt{1+x}\right)^3}{3}+C$

$\sqrt{1+x}=t;\qquad x=t^2-1$

$\frac{1}{2\sqrt{1+x}}dx=dt\qquad dx=2t\,dt$

f) $\int x\sec^2 3x\,dx=\frac{x}{3}\operatorname{tg}3x-\frac{1}{3}\int\operatorname{tg}3x\,dx=\frac{x}{3}\operatorname{tg}3x+\frac{1}{9}\ln\cos 3x+C$

$u=x$	$du=dx$
$dv=\sec^2 3x$	$v=\frac{1}{3}\operatorname{tg}3x$

$\int\operatorname{tg}3x\,dx=\int\frac{\operatorname{sen}3x}{\cos 3x}dx=-\frac{1}{3}\ln\cos x$

g) $\int\operatorname{sen}3x\operatorname{sen}x\,dx=-\operatorname{sen}3x\cos x-3\int\cos 3x\cos x\,dx=-\operatorname{sen}3x\cos x+$

$+3\left(\cos 3x\operatorname{sen}x-3\int\operatorname{sen}3x\operatorname{sen}x\,dx\right)=-\operatorname{sen}3x\cos x+3\cos 3x\ \operatorname{sen}x-9\int\operatorname{sen}3x\operatorname{sen}x\,dx$

$u=\operatorname{sen}3x$	$du=3\cos x\,dx$
$dv=\operatorname{sen}xdx$	$v=-\cos x$

$u=\cos 3x$	$du=-3\operatorname{sen}3x\,dx$
$dv=\cos x\,dx$	$v=\operatorname{sen}x$

Llamamos $I=\int\operatorname{sen}3x\operatorname{sen}x\,dx\Rightarrow\begin{cases}I=-\operatorname{sen}3x\cos x+3\cos 3x\ \operatorname{sen}x+9I\Rightarrow\\ \Rightarrow I=\dfrac{\operatorname{sen}3x\cos x-3\cos 3x\ \operatorname{sen}x}{8}\end{cases}$

h) $\int x\operatorname{arctg}x\,dx=\frac{x^2}{2}\operatorname{arctg}x-\frac{1}{2}\int\frac{x^2}{1+x^2}dx=\frac{x^2}{2}\operatorname{arctg}x-\frac{1}{2}\left(x-\operatorname{arctg}x\right)+C$

$u=\operatorname{arctg}x$	$du=\frac{1}{1+x^2}dx$
$dv=x\,dx$	$v=\frac{x^2}{2}$

$\int\frac{x^2}{1+x^2}dx=\int\left(1-\frac{1}{1+x^2}\right)dx=x-\operatorname{arctg}x$

$$\begin{array}{r|l} x^2 & 1+x^2 \\ \hline -x^2-1 & 1 \\ \hline -1 & \end{array}$$

$D=d\times c+r\Rightarrow\qquad \frac{D}{d}=\frac{\not d\times c}{\not d}+\frac{r}{d}$

i)
$$I=\int e^{nx}\operatorname{sen} bx\,dx=-\frac{\cos bx\,e^{nx}}{b}+\frac{n}{b}\int e^{nx}\cos bx\,dx=-\frac{\cos bx\,e^{nx}}{b}+$$
$$+\frac{n}{b}\left(\frac{\operatorname{sen} bx\,e^{nx}}{b}-\frac{n}{b}\int e^{nx}\operatorname{sen} bx\,dx\right)=-\frac{\cos bx\,e^{nx}}{b}+\frac{n}{b}\frac{\operatorname{sen} bx\,e^{nx}}{b}-\frac{n^2}{b^2}I \Rightarrow$$

$$\left(1+\frac{n^2}{b^2}\right)I=\frac{e^{nx}}{b}\left(\frac{n}{b}\operatorname{sen} x-\cos bx\right)\Rightarrow \frac{b^2+n^2}{b^2}I=\frac{e^{nx}}{b}\left(\frac{n}{b}\operatorname{sen} x-\cos bx\right)\Rightarrow$$

$$\Rightarrow I=\frac{\frac{e^{nx}}{\cancel{b}}\left(\frac{n}{b}\operatorname{sen} x-\cos bx\right)b^{\cancel{2}}}{b^2+n^2}=\frac{b\,e^{nx}}{b^2+n^2}\left(\frac{n}{b}\operatorname{sen} x-\cos bx\right)+C$$

j) $$\int\frac{1+x}{1+x^2}dx=\int\frac{dx}{1+x^2}+\frac{1}{2}\int\frac{2x}{1+x^2}dx=\operatorname{arctg} x+\frac{1}{2}\ln\left(1+x^2\right)+C$$

k) $$\int x\,e^{4x}dx=\frac{x}{4}e^{4x}-\frac{1}{4}\int e^{4x}dx=\frac{x}{4}e^{4x}-\frac{1}{16}e^{4x}+C$$

$$\begin{array}{|ll|}\hline u=x & du=dx\\ dv=e^{4x}dx & v=\frac{1}{4}e^{4x}\\ \hline\end{array}$$

l) $$\int\frac{x}{x^2+9}dx=\frac{1}{2}\int\frac{2x}{x^2+9}dx=\frac{1}{2}\ln\left(x^2+9\right)+C$$

m) $$\int e^{-x}\cos x\,dx=-\cos x\,e^{-x}-\int e^{-x}\operatorname{sen} x\,dx=-\cos x\,e^{-x}+\operatorname{sen} x\,e^{-x}-\int e^{-x}\cos x\,dx$$

$$\begin{array}{|ll|}\hline u=\cos x & du=-\operatorname{sen} x\,dx\\ dv=e^{-x}dx & -e^{-x}\\ \hline\end{array}\quad \begin{array}{|ll|}\hline u=\operatorname{sen} x & du=\cos x\,dx\\ dv=e^{-x}dx & v=-e^{-x}\\ \hline\end{array}$$

Llamamos $I=\int e^{-x}\operatorname{sen} x\,dx\Rightarrow$

$$I=\operatorname{sen} x\,e^{-x}-\cos x\,e^{-x}-I\Rightarrow I=\frac{\operatorname{sen} x\,e^{-x}-\cos x\,e^{-x}}{2}=\frac{e^{-x}}{2}\left(\operatorname{sen} x-\cos x\right)+C$$

n) $$\int x^3e^{-x^2}dx=\frac{1}{2}\int x^3e^{-t}\frac{dt}{x}=\frac{1}{2}\int t\,e^{-t}\,dt=\frac{1}{2}\left(-t\,e^{-t}-\int e^{-t}dt\right)=\frac{1}{2}\left(-x^2e^{-x^2}-e^{-x^2}\right)+C$$

$$\begin{array}{l} x^2=t\\ 2x\,dx=dt\end{array}\quad \begin{array}{|ll|}\hline u=t & du=dt\\ dv=e^{-t}dx & v=-e^{-t}\\ \hline\end{array}$$

o)
$$\int\frac{1+x}{1+\sqrt{x}}dx=2\int\frac{1+t^2}{1+t}\cdot t\cdot dt=2\int\frac{t^3+t}{t+1}dt=2\int\left(t^2-t+2-\frac{2}{t+1}\right)dt=$$
$$2\left(\frac{t^3}{3}-\frac{t^2}{2}+2t-\ln\left(t+1\right)\right)=\frac{2\left(\sqrt{x}\right)^3}{3}-\frac{\left(\sqrt{x}\right)^2}{2}+2\sqrt{x}-\ln\left|\sqrt{x}+1\right|+C$$

$$\begin{array}{l}\sqrt{x}=t;\ x=t^2\\ dx=2t\,dt\end{array}\qquad \begin{array}{r|rrrr} & 1 & 0 & 1 & 0\\ -1 & & -1 & 1 & -2\\ \hline & 1 & -1 & 2 & \underline{|-2}\end{array}$$

p)
$$\int x^2e^{-x}dx=-x^2e^{-x}+2\int x\,e^{-x}dx=-x^2e^{-x}+2\left(-x\,e^{-x}+\int e^{-x}dx\right)=-x^2e^{-x}+2\left(-x\,e^{-x}-e^{-x}\right)=$$
$$=-e^{-x}\left(x^2+2x+2\right)+C$$

$$\begin{array}{|ll|}\hline u=x^2 & du=2x\,dx\\ dv=e^{-x}dx & v=-e^{-x}dx\\ \hline\end{array}\quad \begin{array}{|ll|}\hline u=x & du=dx\\ dv=e^{-x}dx & v=-e^{-x}\\ \hline\end{array}$$

q) $$\int (x^2+1)e^{-2x}dx = -\frac{(x^2+1)e^{-2x}}{2} + \frac{2}{2}\int x e^{-2x}dx = \frac{(x^2+1)e^{-2x}}{2} - \frac{1}{2}x e^{-2x} +$$
$$+\frac{1}{2}\int e^{-2x}dx = \frac{(x^2+1)e^{-2x}}{2} - \frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x} + C$$

$u = x^2+1$	$du = 2x\,dx$
$dv = e^{-2x}dx$	$v = -\frac{1}{2}e^{-2x}$

$u = x$	$du = dx$
$dv = e^{-2x}dx$	$v = -\frac{1}{2}e^{-2x}$

r) $$\int \frac{dx}{\sqrt{x}\cos^2\sqrt{x}} = 2\int \frac{\cancel{t}\,dt}{\cancel{t}\cos^2 t} = 2\int \frac{dt}{\cos^2 t} = 2\int \sec^2 t\,dt = 2\,\mathrm{tg}\,t = 2\,\mathrm{tg}\sqrt{x} + C$$

$$t = \sqrt{x}$$
$$dt = \frac{1}{2\sqrt{x}}dx; \quad dx = 2\sqrt{x}\,dx$$

s) $$\int x\left[\mathrm{sen}\,2x + \ln(1+x^2)\right]dx = I_1 + I_2 =$$
$$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\mathrm{sen}\,2x + \frac{1}{2}(x^2+1)\ln(x^2+1) - \frac{1}{2}(x^2+1) + C$$

$$I_1 = \int x\,\mathrm{sen}\,2x\,dx = -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx = -\frac{1}{2}x\cos 2x + \frac{1}{4}\mathrm{sen}\,2x$$

$u = x$	$du = dx$
$dv = \mathrm{sen}\,2x\,dx$	$v = -\frac{1}{2}\cos 2x$

$$I_2 = \int x\ln(x^2+1)dx = \frac{1}{2}\int \cancel{x}\ln t\frac{dt}{\cancel{x}} = \frac{1}{2}\int \ln t\,dt = \frac{1}{2}(t\ln t - t) =$$
$$= \frac{1}{2}\left[(x^2+1)\ln(x^2+1) - (x^2+1)\right]$$

$$t = 1 + x^2$$
$$dt = 2x\,dx$$

$u = \ln t$	$du = \frac{1}{t}dt$
$dv = dt$	$v = t$