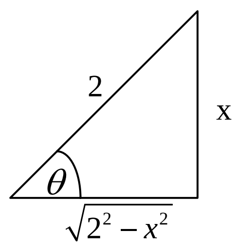
INTEGRACION POR SUSTITUCION TRIGONOMETRICA

Existen integrales que contienen expresiones de las formas: $a^2 - x^2, a^2 + x^2$ $x^2 - a^2$, las que tienen fácil solución si se hace la sustitución trigonométrica adecuada. A saber, si la expresión es: $a^2 - x^2$, la sustitución adecuada es: $x = a\,\text{sen}\,\theta$ ó $x = a\cos\theta$. Si la expresión es: $a^2 + x^2$, entonces: $x = a\sec\theta$

EJERCICIOS DESARROLLADOS

1. **Encontrar:** $\int \frac{dx}{\sqrt{(4-x^2)^3}}$

Solución.- Dada le expresión: $4 - x^2$, la forma es: $a^2 - x^2$, la sustitución adecuada es: $x = a\,\text{sen}\,\theta$ o sea: $x = 2\,\text{sen}\,\theta \therefore dx = 2\cos\theta d\theta$. Además: $\text{sen}\,\theta = \frac{x}{a}$. Una figura auxiliar adecuada para ésta situación, es:



$$\int \frac{dx}{\sqrt{(4-x^2)^3}} = \int \frac{dx}{\sqrt{(2^2-x^2)^3}} = \int \frac{2\cos\theta d\theta}{\sqrt{(2^2-2^2\,\text{sen}^2\,\theta)^3}} = \int \frac{2\cos\theta d\theta}{\sqrt{\left[(2^2(1-\text{sen}^2\,\theta)\right]^3}}$$

$$= \int \frac{2\cos\theta d\theta}{\sqrt{(2^2\cos^2\theta)^3}} = \int \frac{2\cos\theta d\theta}{(2\cos\theta)^3} = \int \frac{2\cos\theta d\theta}{2^3\cos^3\theta} = \frac{1}{2^2}\int \frac{d\theta}{\cos^2\theta} = \frac{1}{4}\int \sec^2\theta d\theta$$

$= \frac{1}{4}\int \sec^2\theta d\theta = \frac{1}{4}\tau g\theta + c$. A partir de la figura triangular se tiene:

$\tau g\theta = \frac{x}{\sqrt{4-x^2}}$, Luego: $\frac{1}{4}\tau g\theta + c = \frac{1}{4}\frac{x}{\sqrt{4-x^2}} + c$

Respuesta: $\int \frac{dx}{\sqrt{(4-x^2)^3}} = \frac{1}{4}\frac{x}{\sqrt{4-x^2}} + c$

6.2.-Encontrar: $\int \frac{\sqrt{25-x^2}}{x}dx$

Solución.-

$$\int \frac{\sqrt{25-x^2}}{x}dx = \int \frac{\sqrt{5^2-x^2}}{x}dx$$, la forma es: a^2-x^2 , luego:

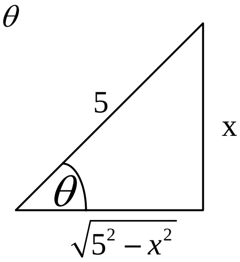
Sea: $x = 5\,\text{sen}\,\theta \therefore dx = 5\cos\theta d\theta$, $\sqrt{5^2-x^2} = 5\cos\theta$

Además: $\text{sen}\,\theta = \frac{x}{5}$

$$\int \frac{\sqrt{5^2-x^2}}{x}dx = \int \frac{\not{5}\cos\theta 5\cos\theta d\theta}{\not{5}\,\text{sen}\,\theta} = 5\int \frac{\cos^2\theta d\theta}{\text{sen}\,\theta} = 5\int \frac{(1-\text{sen}^2\,\theta)d\theta}{\text{sen}\,\theta}$$

$$= 5\int \frac{d\theta}{\text{sen}\,\theta} - 5\int \text{sen}\,\theta d\theta = 5\int \cos ec\theta - 5\int \text{sen}\,\theta d\theta$$

$$= 5\ell\eta\left|\cos ec\theta - \text{co}\,\tau g\theta\right| + 5\cos\theta + c\ .$$



De la figura se tiene:

$\cos ec\theta = \frac{5}{x}, \text{co}\,\tau g\theta = \frac{\sqrt{25-x^2}}{x}$, luego:

$$= 5\ell\eta\left|\frac{5}{x} - \frac{\sqrt{25-x^2}}{x}\right| + \not{5}\frac{\sqrt{25-x^2}}{\not{5}} + c = 5\ell\eta\left|\frac{5-\sqrt{25-x^2}}{x}\right| + \sqrt{25-x^2} + c$$

Respuesta: $\int \frac{\sqrt{25-x^2}}{x}dx = 5\ell\eta\left|\frac{5-\sqrt{25-x^2}}{x}\right| + \sqrt{25-x^2} + c$

6.3.-Encontrar: $\int \frac{dx}{\sqrt{(4x-x^2)^3}}$

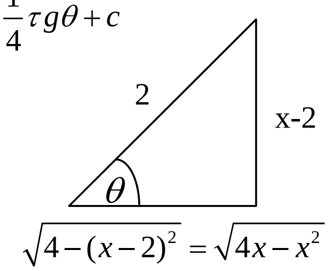
Solución.- $4x-x^2 = -(x^2-4x) = -(x^2-4x+4-4) = 4-(x^2-4x+4) = 2^2-(x-2)^2$

$\int \frac{dx}{\sqrt{(4x-x^2)^3}} = \int \frac{dx}{(\sqrt{2^2-(x-2)^2})^3}$, la forma es: a^2-u^2 ,

Luego: $x-2 = 2\,\text{sen}\,\theta \therefore dx = 2\cos\theta d\theta$, $\sqrt{2^2-(x-2)^2} = 2\cos\theta$

Además: $\text{sen}\,\theta = \frac{x-2}{2}$

$$\int \frac{dx}{(\sqrt{2^2-(x-2)^2})^3} = \int \frac{2\cos\theta d\theta}{2^3\cos^3\theta} = \frac{1}{4}\int \frac{d\theta}{\cos^2\theta} = \frac{1}{4}\int \sec^2\theta d\theta = \frac{1}{4}\tau g\theta + c$$



De la figura se tiene:

Pero: $\tau g\theta = \frac{x-2}{\sqrt{4x-x^2}}$, luego: $\frac{1}{4}\tau g\theta + c = \frac{x-2}{4\sqrt{4x-x^2}} + c$

Respuesta: $\int \frac{dx}{\sqrt{(4x-x^2)^3}} = \frac{x-2}{4\sqrt{4x-x^2}} + c$

6.4.-Encontrar: $\int \frac{x^2dx}{(a^2-x^2)^{3/2}}$

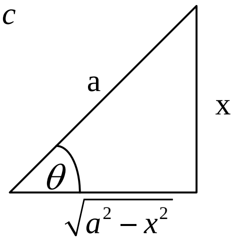
Solución.-

$\int \frac{x^2dx}{(a^2-x^2)^{3/2}} = \int \frac{x^2dx}{(\sqrt{a^2-x^2})^3}$, la forma es: a^2-x^2

Luego: $x = a\,\mathrm{sen}\,\theta, dx = a\cos\theta, \sqrt{a^2-x^2} = a\cos\theta$, además: $\mathrm{sen}\,\theta = \frac{x}{a}$

$$\int \frac{x^2dx}{(\sqrt{a^2-x^2})^3} = \int \frac{a^2\,\mathrm{sen}^2\,\theta a\cos\theta d\theta}{(a\cos\theta)^3} = \int \frac{\cancel{a^3}\,\mathrm{sen}^2\,\theta \cancel{\cos\theta} d\theta}{\cancel{a^3}\,\cancel{\cos\theta}\cos^2\theta} = \int \frac{\mathrm{sen}^2\,\theta d\theta}{\cos^2\theta}$$

$$= \int \frac{(1-\cos^2\theta)d\theta}{\cos^2\theta} = \int \frac{d\theta}{\cos^2\theta} - \int d\theta = \int \sec^2\theta d\theta - \int d\theta = \tau g\theta - \theta + c$$



De la figura se tiene:

Pero: $\tau g\theta = \frac{x}{\sqrt{a^2-x^2}}$, además: $\mathrm{sen}\,\theta = \frac{x}{a}$ y $\theta = \mathrm{arcsen}\frac{x}{a}$

Luego: $\tau g\theta - \theta + c = \frac{x}{\sqrt{a^2-x^2}} - \mathrm{arcsen}\frac{x}{a} + c$

Respuesta: $\int \frac{x^2dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{\sqrt{a^2-x^2}} - \mathrm{arcsen}\frac{x}{a} + c$

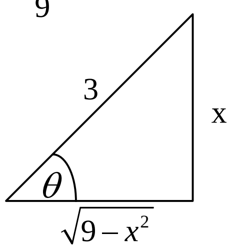
6.5.-Encontrar: $\int \frac{dx}{x^2\sqrt{9-x^2}}$

Solución.-

$\int \frac{dx}{x^2\sqrt{9-x^2}} = \int \frac{dx}{x^2\sqrt{3^2-x^2}}$, la forma es: a^2-x^2

Luego: $x = 3\,\mathrm{sen}\,\theta, dx = 3\cos\theta d\theta, \sqrt{3^2-x^2} = 3\cos\theta$, además: $\mathrm{sen}\,\theta = \frac{x}{3}$

$$\int \frac{dx}{x^2\sqrt{3^2-x^2}} = \int \frac{\cancel{3\cos\theta} d\theta}{3^2\,\mathrm{sen}^2\,\theta \cancel{3\cos\theta}} = \frac{1}{9}\int \frac{d\theta}{\mathrm{sen}^2\,\theta} = \frac{1}{9}\int \mathrm{cosec}^2\theta d\theta = -\frac{1}{9}\mathrm{co}\tau g\theta + c$$



De la figura se tiene:

Pero: $\cot g\theta = \frac{\sqrt{9-x^2}}{x}$, luego: $\frac{1}{9}\cot g\theta + c = -\frac{\sqrt{9-x^2}}{9x} + c$

Respuesta: $\int \frac{dx}{x^2\sqrt{9-x^2}} = -\frac{\sqrt{9-x^2}}{9x} + c$

6.6.-Encontrar: $\int \frac{x^2dx}{\sqrt{9-x^2}}$

Solución.-

$\int \frac{x^2dx}{\sqrt{9-x^2}} = \int \frac{x^2dx}{\sqrt{3^2-x^2}}$, la forma es: $a^2 - x^2$

Luego: $x = 3\operatorname{sen}\theta, dx = 3\cos\theta d\theta, \sqrt{3^2-x^2} = 3\cos\theta$, además: $\operatorname{sen}\theta = \frac{x}{3}$

Usaremos la misma figura anterior, luego:

$$\int \frac{x^2dx}{\sqrt{3^2-x^2}} = \int \frac{3^2\operatorname{sen}^2\theta \cancel{3\cos\theta} d\theta}{\cancel{3\cos\theta}} = 9\int \operatorname{sen}^2\theta d\theta = 9\int \frac{(1-\cos 2\theta)d\theta}{2}$$

$$\frac{9}{2}\int \theta - \frac{9}{2}\int \cos 2\theta d\theta = \frac{9}{2}\theta - \frac{9}{4}\operatorname{sen} 2\theta + c = \frac{9}{2}\theta - \frac{9}{4}2\operatorname{sen}\theta\cos\theta + c$$

$= \frac{9}{2}\theta - \frac{9}{2}\operatorname{sen}\theta\cos\theta + c$, de la figura se tiene que: $\operatorname{sen}\theta = \frac{x}{3}, \cos\theta = \frac{\sqrt{9-x^2}}{3}$ y $\theta = \operatorname{arcsen}\frac{x}{3}$, luego es equivalente:

$$= \frac{9}{2}\operatorname{arcsen}\frac{x}{3} - \frac{9}{4}\frac{x}{3}\frac{\sqrt{9-x^2}}{3} + c = \frac{9}{2}\left(\operatorname{arcsen}\frac{x}{3} - \frac{\sqrt{9-x^2}}{9}\right) + c$$

Respuesta: $\int \frac{x^2dx}{\sqrt{9-x^2}} = \frac{9}{2}\left(\operatorname{arcsen}\frac{x}{3} - \frac{\sqrt{9-x^2}}{9}\right) + c$

6.7.-Encontrar: $\int \sqrt{x^2-4}dx$

Solución.-

$\int \sqrt{x^2-4}dx = \int \sqrt{x^2-2^2}dx$, la forma es: $x^2 - a^2$

Luego: $x = 2\sec\theta, dx = 2\sec\theta\tau g\theta d\theta, \sqrt{x^2-2^2} = 2\tau g\theta$, además: $\sec\theta = \frac{x}{2}$

$$\int \sqrt{x^2-2^2}dx = \int 2\tau g\theta 2\sec\theta\tau g\theta d\theta = 4\int \sec\theta\tau g^2\theta d\theta = 4\int \sec\theta(\sec^2\theta - 1)d\theta$$

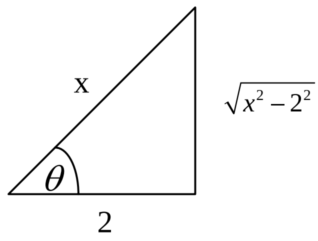
$$= 4\int \sec^3\theta d\theta - 4\int \sec\theta d\theta$$

Se sabe que: $\int \sec^3\theta d\theta = \frac{\sec\theta\tau g\theta}{2} + \frac{1}{2}\ell\eta\left|\sec\theta + \tau g\theta\right| + c$, luego lo anterior es equivalente a:

$$= 4\left(\frac{1}{2}\sec\theta\tau g\theta + \frac{1}{2}\ell\eta\left|\sec\theta + \tau g\theta\right|\right) - 4\ell\eta\left|\sec\theta + \tau g\theta\right| + c$$

$$= 2\sec\theta\tau g\theta + 2\ell\eta\left|\sec\theta + \tau g\theta\right| - 4\ell\eta\left|\sec\theta + \tau g\theta\right| + c$$

$$= 2\sec\theta\tau g\theta - 2\ell\eta\left|\sec\theta + \tau g\theta\right| + c$$



De la figura se tiene:

$\sec\theta = \frac{x}{2}, \tau g\theta = \frac{\sqrt{x^2-4}}{2}$, luego:

$$= \cancel{2}\frac{x}{\cancel{2}}\frac{\sqrt{x^2-4}}{2} - 2\ell\eta\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + c = \frac{x\sqrt{x^2-4}}{2} - 2\ell\eta\left|\frac{x+\sqrt{x^2-4}}{2}\right| + c$$

$$= \frac{x\sqrt{x^2-4}}{2} - 2\ell\eta\left|x + \sqrt{x^2-4}\right| - 2\ell\eta 2 + c$$

Respuesta: $\int\sqrt{x^2-4}\,dx = \frac{x\sqrt{x^2-4}}{2} - 2\ell\eta\left|x+\sqrt{x^2-4}\right| + c$

6.8.-Encontrar: $\int\frac{x^2dx}{\sqrt{x^2-16}}$

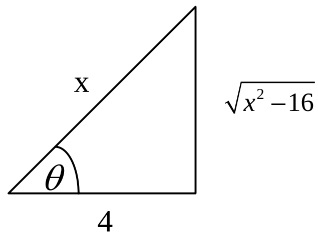
Solución.-

$\int\frac{x^2dx}{\sqrt{x^2-16}} = \int\frac{x^2dx}{\sqrt{x^2-4^2}}$, la forma es: $x^2 - a^2$

Luego: $x = 4\sec t, dx = 4\sec t\tau gt dt, \sqrt{x^2-4^2} = 4\tau gt$, además: $\sec t = \frac{x}{4}$

$$\int\frac{x^2dx}{\sqrt{x^2-4^2}} = \int\frac{4^2\sec^2 t(\cancel{4}\sec t\,\cancel{\tau gt}\,dt)}{\cancel{4}\,\cancel{\tau gt}} = 16\int\sec^3 tdt$$

$$= 16\left(\frac{1}{2}\sec t\tau gt + \frac{1}{2}\ell\eta\left|\sec t + \tau gt\right| + c\right) = 8\sec t\tau gt + 8\ell\eta\left|\sec t + \tau gt\right| + c$$



De la figura se tiene:

$\sec t = \frac{x}{4}, \tau gt = \frac{\sqrt{x^2-16}}{4}$, luego equivale a:

$$= 8\frac{x}{4}\frac{\sqrt{x^2-16}}{4} + 8\ell\eta\left|\frac{x}{4} + \frac{\sqrt{x^2-16}}{4}\right| + c = \frac{x}{2}\sqrt{x^2-16} + 8\ell\eta\left|\frac{x\sqrt{x^2-16}}{4}\right| + c$$

$$= \frac{x}{2}\sqrt{x^2-16} + 8\ell\eta\left|x\sqrt{x^2-16}\right| - 8\ell\eta 4 + c = \frac{x}{2}\sqrt{x^2-16} + 8\ell\eta\left|x\sqrt{x^2-16}\right| + c$$

Respuesta: $\int \frac{x^2dx}{\sqrt{x^2-16}} = \frac{x}{2}\sqrt{x^2-16} + 8\ell\eta\left|x\sqrt{x^2-16}\right| + c$

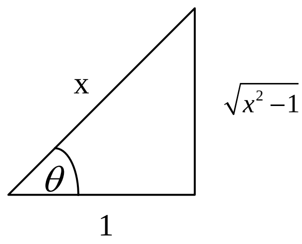
6.9.-Encontrar: $\int \frac{dx}{x\sqrt{x^2-1}}$

Solución.-

$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{dx}{x\sqrt{x^2-1^2}}$, la forma es: x^2-a^2

Luego: $x = \sec t, dx = \sec t\tau gtdt, \sqrt{x^2-1^2} = \tau gt$, además:

$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\cancel{\sec t\tau gt}\,dt}{\cancel{\sec t\tau gt}} = \int dt = t + c$,



De la figura se tiene:

Dado que: $\sec t = x \Rightarrow t = \operatorname{arc}\sec x$, luego:

$t + c = \operatorname{arc}\sec x + c$

Respuesta: $\int \frac{dx}{x\sqrt{x^2-1}} = \operatorname{arc}\sec x + c$

6.10.-Encontrar: $\int \frac{dx}{(\sqrt{4x^2-24x+27})^3}$

Solución.-

$\int \frac{dx}{(\sqrt{4x^2-24x+27})^3} = \int \frac{dx}{\sqrt{4(x^2-6x+27/4)^3}} = \int \frac{dx}{\sqrt{4^3}\left(\sqrt{x^2-6x+27/4}\right)^3}$

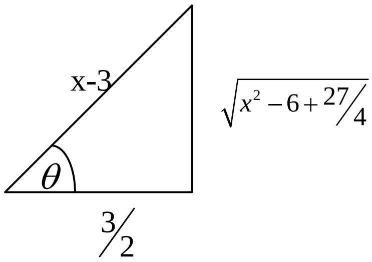
$= \frac{1}{8}\int \frac{dx}{\sqrt{(x^2-6x+27/4)^3}}$, Se tiene:

$x^2-6x+\frac{27}{4} = (x^2-6x+\_\_) + \frac{27}{4} - \_\_ = (x^2-6x+9) + \frac{27}{4} - 9$

$= (x^2-6x+9) - \frac{9}{4} = (x^2-6x+27/4) = (x-3)^2 - (3/2)^2$, la expresión anterior equivale a:

$\frac{1}{8}\int \frac{dx}{(\sqrt{x^2-6x+27/4})^3} = \frac{1}{8}\int \frac{dx}{\left[\sqrt{(x-3)^2-(3/2)^2}\right]^3}$, siendo la forma: u^2-a^2, luego:

$x-3 = 3/2\sec t, dx = 3/2\sec t\tau gtdt$, además: $\sec t = \frac{x-3}{3/2}$



De la figura se tiene:

$\sec t = \frac{x}{4}, \tau gt = \frac{\sqrt{x^2-16}}{4}$, luego equivale a:

$$\frac{1}{8}\int\frac{dx}{\left[\sqrt{(x-3)^2-(3/2)^2}\right]^3} = \frac{1}{8}\int\frac{3/2\sec t\tau gtdt}{(3/2)^2\tau g^3 t} = \frac{1}{8}\frac{1}{\frac{3^2}{2^2}}\int\frac{\sec tdt}{\tau g^2 t} = \frac{1}{18}\int\frac{\frac{1}{\cos t}}{\frac{\operatorname{sen}^2 t}{\cos^2 t}}$$

$$= \frac{1}{18}\int\frac{\cos tdt}{(\operatorname{sen} t)^2} = \frac{1}{18}\int(\operatorname{sen} t)^{-2}\cos tdt = \frac{1}{18}\frac{(\operatorname{sen} t)^{-1}}{-1} + c = -\frac{1}{18}\frac{1}{(\operatorname{sen} t)} + c$$

$= -\frac{1}{18}\operatorname{cos}ect + c$, como: $\operatorname{cos}ect = \frac{x-3}{\sqrt{x^2-6x+27/4}}$, entonces:

$$= -\frac{1}{18}\frac{x-3}{\sqrt{x^2-6x+27/4}} + c = -\frac{1}{18}\frac{x-3}{\sqrt{\frac{4x^2-24x+27}{4}}} + c = -\frac{1}{18}\frac{x-3}{\frac{\sqrt{4x^2-24x+27}}{2}} + c$$

$$= -\frac{1}{9}\frac{x-3}{\sqrt{4x^2-24x+27}} + c$$

Respuesta: $\int\frac{dx}{(\sqrt{4x^2-24x+27})^3} = -\frac{1}{9}\frac{x-3}{\sqrt{4x^2-24x+27}} + c$

6.11.-Encontrar: $\int\frac{dx}{\sqrt{(16+x^2)^4}}$

Solución.-

$$\int\frac{dx}{\sqrt{(16+x^2)^4}} = \int\frac{dx}{\sqrt{(4^2+x^2)^4}}$$

Luego: $x = 4\tau gt, dx = 4\sec^2 tdt, \sqrt{4^2+x^2} = 4\sec t$, además: $\tau gt = x/4$

$$\int\frac{dx}{\sqrt{(4^2+x^2)^4}} = \int\frac{4\sec^2 tdt}{4^4\sec^4 t} = \frac{1}{64}\int\frac{dt}{\sec^2 t} = \frac{1}{64}\int\cos^2 tdt = \frac{1}{64}\int\frac{(1+\cos 2t)}{2}dt$$

$$= \frac{1}{128}\int dt + \frac{1}{128}\int\cos 2tdt = \frac{1}{128}t + \frac{1}{256}\operatorname{sen} 2t + c$$

Como: $\tau gt = x/4 \Rightarrow t = \operatorname{arc}\tau g\, x/4$, $\operatorname{sen} 2t = 2\operatorname{sen} t\cos t$; luego:

$\frac{1}{128}t + \frac{1}{256}\operatorname{sen} 2t + c = 2\frac{x}{\sqrt{16+x^2}}\frac{4}{\sqrt{16+x^2}} = \frac{8x}{16+x^2}$, Se tiene:

$$\frac{1}{128}\operatorname{arc}\tau g\, x/4 + \frac{1}{256}\frac{8x}{16+x^2} + c = \frac{1}{128}\operatorname{arc}\tau g\, x/4 + \frac{x}{32(16+x^2)} + c$$

Respuesta: $\int \frac{dx}{\sqrt{(16+x^2)^4}} = \frac{1}{128}\operatorname{arc}\tau g\frac{x}{4} + \frac{x}{32(16+x^2)} + c$

6.12.-Encontrar: $\int \frac{x^2dx}{(x^2+100)^{3/2}}$

Solución.-

$$\int \frac{x^2dx}{(x^2+100)^{3/2}} = \int \frac{x^2dx}{(\sqrt{x^2+10^2})^3},$$

se tiene: $x = 10\tau gt, dt = 10\sec^2 tdt$, $\sqrt{x^2+10^2} = 10\sec t$;además: $\tau gt = \frac{x}{10}$, luego:

$$\int \frac{x^2dx}{(\sqrt{x^2+10^2})^3} = \int \frac{10^2\tau g^2t(10\sec^2 t)dt}{(10^3\sec^3 t)} = \int \frac{\tau g^2tdt}{\sec t} = \int \frac{\frac{\operatorname{sen}^2 t}{\cos^2 t}}{\frac{1}{\cos t}}dt = \int \frac{\operatorname{sen}^2 t}{\cos t}dt$$

$$= \int \frac{(1-\cos^2 t)}{\cos t}dt = \int \frac{dt}{\cos t} - \int \cos tdt = \int \sec tdt - \int \cos tdt = \ell\eta\left|\sec t + \tau gt\right| - \operatorname{sen} t + c$$

Como: $\sec t = \frac{\sqrt{100+x^2}}{10}, \tau gt = \frac{x}{10}$, además: $\operatorname{sen} t = \frac{x}{\sqrt{100+x^2}}$

$$= \ell\eta\left|\frac{\sqrt{100+x^2}}{10} + \frac{x}{10}\right| - \frac{x}{\sqrt{x^2+100}} + c = \ell\eta\left|\frac{\sqrt{100+x^2}+x}{10}\right| - \frac{x}{\sqrt{x^2+100}} + c$$

$$= \ell\eta\left|\sqrt{100+x^2}+x\right| - \frac{x}{\sqrt{x^2+100}} - \ell\eta 10 + c = \ell\eta\left|\sqrt{100+x^2}+x\right| - \frac{x}{\sqrt{x^2+100}} + c$$

Respuesta: $\int \frac{x^2dx}{(x^2+100)^{3/2}} = \ell\eta\left|\sqrt{100+x^2}+x\right| - \frac{x}{\sqrt{x^2+100}} + c$

Nota: En los ejercicios 6.11 y 6.12 se ha omitido la figura (triángulo rectángulo). Conviene hacerla y ubicar los datos pertinentes. En adelante se entenderá que el estudiante agregará este complemento tan importante.

6.13.-Encontrar: $\int \frac{x^2dx}{(x^2+8^2)^{3/2}}$

Solución.-

$$\int \frac{x^2dx}{(x^2+8^2)^{3/2}} = \int \frac{x^2dx}{(\sqrt{x^2+8^2})^3},$$

se tiene: $x = 8\tau gt, dt = 8\sec^2 tdt$, $\sqrt{x^2+8^2} = 8\sec t$ además: $\tau gt = \frac{x}{8}$, luego:

$$\int \frac{x^2dx}{(\sqrt{x^2+8^2})^3} = \int \frac{8^2\tau g^2t(8\sec^2 t)}{8^3\sec^3 t}dt = \int \frac{\tau g^2t}{\sec t}dt = \int \sec tdt - \int \cos tdt$$

$= \ell\eta\left|\sec t + \tau gt\right| - \operatorname{sen} t + c$, como: $\sec t = \frac{\sqrt{x^2+64}}{8}, \tau gt = \frac{x}{8}, \operatorname{sen} t = \frac{x}{\sqrt{x^2+64}}$

Se tiene como expresión equivalente:

$$= \ell\eta\left|\frac{\sqrt{x^2+64}}{8} + \frac{x}{8}\right| - \frac{x}{\sqrt{x^2+64}} + c = \ell\eta\left|\frac{\sqrt{x^2+64}+x}{8}\right| - \frac{x}{\sqrt{x^2+64}} + c$$

$$= \ell\eta\left|\sqrt{x^2+64}+x\right| - \frac{x}{\sqrt{x^2+64}} + c$$

Respuesta: $\int \frac{x^2 dx}{(x^2+8^2)^{3/2}} = \ell\eta\left|\sqrt{x^2+64}+x\right| - \frac{x}{\sqrt{x^2+64}} + c$

6.14.-Encontrar: $\int \frac{dx}{(\sqrt{3^2+x^2})^4}$

Solución.- se tiene: $x = 3\tau gt, dx = 3\sec^2 tdt$, $\sqrt{3^2+x^2} = 3\sec t$, además: $\tau gt = \frac{x}{3}$

$$\int \frac{dx}{(\sqrt{3^2+x^2})^4} = \int \frac{\cancel{3}\cancel{\sec^2} t\,dt}{3^{\cancel{4}} + \sec^{\cancel{4}} t} = \frac{1}{3^3}\int \frac{dt}{\sec^2 t} = \frac{1}{27}\int \cos^2 tdt = \frac{1}{54}t + \frac{1}{54}\int \cos 2tdt$$

$$= \frac{1}{54}t + \frac{1}{108}\operatorname{sen} 2t + c_1 = \frac{1}{54}t + \frac{1}{108}2\operatorname{sen} t\cos t + c = \frac{1}{54}t + \frac{1}{54}\operatorname{sen} t\cos t + c$$

Como: $\underline{\tau gt = \frac{x}{3} \Rightarrow t = \operatorname{arc}\tau g\frac{x}{3}}$, además: $\operatorname{sen} t = \frac{x}{\sqrt{9+x^2}}$, $\cos t = \frac{3}{\sqrt{9+x^2}}$

$$= \frac{1}{54}\operatorname{arc}\tau g\frac{x}{3} + \frac{1}{54}\frac{x}{\sqrt{9+x^2}}\frac{3}{\sqrt{9+x^2}} + c = \frac{1}{54}\operatorname{arc}\tau g\frac{x}{3} + \frac{x}{18(9+x^2)} + c$$

Respuesta: $\int \frac{dx}{(\sqrt{3^2+x^2})^4} = \frac{1}{54}\operatorname{arc}\tau g\frac{x}{3} + \frac{x}{18(9+x^2)} + c$

6.15.-Encontrar: $\int \frac{dx}{\sqrt{x^2-4x+13}}$

Solución.- Completando cuadrados se tiene:

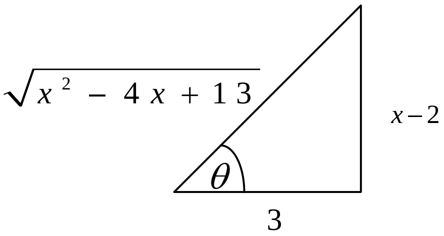
$x^2-4x+13 = (x^2-4x+\_\_) + 13 - \_\_ = (x^2-4x+4) + 13 - 4 = (x-2)^2 + 3^2$

Se tiene: $x-2 = 3\tau gt, dx = 3\sec^2 tdt$, $\sqrt{3^2+x^2} = 3\sec t$

$\sqrt{(x-2)^2+3^2} = \sqrt{x^2-4x+13} = 3\sec t$,

Sea: $x-2 = 3\tau gt, dx = 3\sec^2 tdt$;además: $\tau gt = \frac{x-2}{3}$, luego:

$$\int \frac{dx}{\sqrt{(x-2)^2+3^2}} = \int \frac{\cancel{3}\sec^{\cancel{2}} tdt}{\cancel{3\sec} t} = \int \sec tdt = \ell\eta\left|\sec t + \tau gt\right| + c$$



De la figura se tiene:

$\sec t=\dfrac{\sqrt{x^2-4x+13}}{3}$**,** $\tau gt=\dfrac{x-2}{3}$, luego:

$=\ell\eta\left|\dfrac{\sqrt{x^2-4x+13}}{3}+\dfrac{x-2}{3}\right|+c=\ell\eta\left|\dfrac{\sqrt{x^2-4x+13}+(x-2)}{3}\right|+c$

$=\ell\eta\left|\sqrt{x^2-4x+13}+(x-2)\right|+c$

Respuesta: $\displaystyle\int\frac{dx}{\sqrt{x^2-4x+13}}=\ell\eta\left|\sqrt{x^2-4x+13}+(x-2)\right|+c$

6.16.-Encontrar: $\int\sqrt{1+4x^2}\,dx$

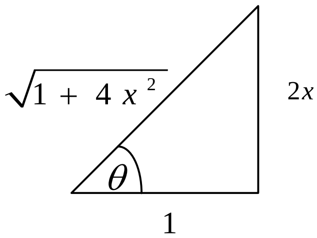
Solución.-

$\int\sqrt{1+4x^2}\,dx=\int\sqrt{1^2+(2x)^2}\,dx$

Se tiene: $2x=\tau gt, 2dx=\sec^2 tdt\Rightarrow dx=\dfrac{1}{2}\sec^2 tdt$, Además: $\tau gt=\dfrac{2x}{1}$

$\displaystyle\int\sqrt{1^2+(2x)^2}\,dx=\int\sqrt{1^2+\tau g^2t}\,\frac{1}{2}\sec^2 dt=\frac{1}{2}\int\sec t\sec^2 tdt=\frac{1}{2}\int\sec^3 tdt$

$=\dfrac{1}{4}\sec t\tau gt+\dfrac{1}{4}\ell\eta\left|\sec t\tau gt\right|+c$,



De la figura se tiene:

$\sec t=\dfrac{\sqrt{1+4x^2}}{1}$, $\tau gt=2x$

$=\dfrac{1}{4}\sqrt{1+4x^2}\,2x+\dfrac{1}{4}\ell\eta\left|\sqrt{1+4x^2}\right|+2x+c$

Respuesta: $\displaystyle\int\sqrt{1+4x^2}\,dx=\frac{1}{4}\sqrt{1+4x^2}\,2x+\frac{1}{4}\ell\eta\left|\sqrt{1+4x^2}\right|+2x+c$

EJERCICIOS PROPUESTOS:

Utilizando esencialmente la técnica de sustitución por variables trigonométricas, encontrar las integrales siguientes:

6.17.- $\int\sqrt{4-x^2}$ **6.18.-** $\displaystyle\int\frac{dx}{\sqrt{a^2-x^2}}$ **6.19.-** $\displaystyle\int\frac{dx}{x^2+a^2}$

6.20.- $\int \frac{dx}{x^2-a^2}$

6.21.- $\int \frac{dx}{\sqrt{x^2+a^2}}$

6.22.- $\int \frac{dx}{\sqrt{x^2-a^2}}$

6.23.- $\int \frac{dx}{x\sqrt{x^2-9}}$

6.24.- $\int \frac{dx}{x\sqrt{x^2-2}}$

6.25.- $\int \frac{dx}{x\sqrt{1+x^2}}$

6.26.- $\int \frac{x^2dx}{\sqrt{1-x^2}}$

6.27.- $\int \frac{x^3dx}{\sqrt{2-x^2}}$

6.28.- $\int \frac{\sqrt{x^2-9}}{x}dx$

6.29.- $\int \frac{dx}{x\sqrt{4x^2-16}}$

6.30.- $\int \frac{\sqrt{x^2+1}}{x}dx$

6.31.- $\int \frac{dx}{x^2\sqrt{4-x^2}}$

6.32.- $\int \sqrt{a-x^2}\,dx$

6.33.- $\int \sqrt{a^2-x^2}\,dx$

6.34.- $\int \frac{x^2dx}{\sqrt{x^2+a^2}}$

6.35.- $\int \frac{dx}{x^2\sqrt{x^2+9}}$

6.36.- $\int \frac{dx}{\sqrt{5-4x^2}}$

6.37.- $\int \frac{x^2dx}{(4-x^2)^{3/2}}$

6.38.- $\int x^2\sqrt{5-x^2}\,dx$

6.39.- $\int \frac{dx}{x^4\sqrt{x^2+3}}$

6.40.- $\int x^3\sqrt{a^2x^2+b^2}\,dx$

6.41.- $\int \frac{dx}{x^2\sqrt{x^2+a^2}}$

6.42.- $\int \frac{dx}{(x^2+a^2)^2}$

6.43.- $\int x^3\sqrt{a^2x^2-b^2}\,dx$

6.44.- $\int \frac{dx}{x^2\sqrt{a^2-x^2}}$

6.45.- $\int \frac{\sqrt{2x^2-5}}{x}dx$

6.46.- $\int \frac{x^3dx}{\sqrt{3x^2-5}}$

6.47.- $\int \frac{\sqrt{x^2-100}}{x}dx$

6.48.- $\int \frac{dx}{x^2\sqrt{x^2-2}}$

6.49.- $\int \frac{dx}{x\sqrt{9-x^2}}$

6.50.- $\int \frac{\sqrt{x^2+a^2}}{x}dx$

6.51.- $\int \frac{xdx}{\sqrt{a^2-x^2}}$

6.52.- $\int \frac{dx}{\sqrt{1-4x^2}}$

6.53.- $\int \frac{dx}{\sqrt{4+x^2}}$

6.54.- $\int \frac{xdx}{\sqrt{4+x^2}}$

6.55.- $\int \frac{dx}{x\sqrt{a^2+x^2}}$

6.56.- $\int \frac{(x+1)dx}{\sqrt{4-x^2}}$

6.57.- $\int \frac{dx}{\sqrt{2-5x^2}}$

6.58.- $\int \frac{dx}{(a^2-x^2)^{3/2}}$

6.59.- $\int \frac{dx}{\sqrt{4-(x-1)^2}}$

6.60.- $\int \frac{x^2dx}{\sqrt{2x-x^2}}$

6.61.- $\int \frac{x^2dx}{\sqrt{17-x^2}}$

6.62.- $\int \frac{x^2dx}{\sqrt{21+4x-x^2}}$

6.63.- $\int \frac{dx}{(x^2-2x+5)^{3/2}}$

6.64.- $\int \frac{(2x+1)dx}{\sqrt{(4x^2-2x+1)^3}}$

6.65.- $\int \frac{dx}{(x-1)\sqrt{x^2-3x+2}}$

6.66.- $\int \frac{xdx}{\sqrt{x^2-2x+5}}$

6.67.- $\int \frac{(x+1)dx}{\sqrt{2x-x^2}}$

6.68.- $\int \frac{(x-1)dx}{\sqrt{x^2-4x+3}}$

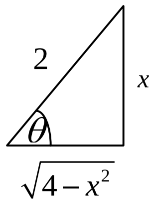
6.69.- $\int \frac{dx}{\sqrt{x^2-2x-8}}$

6.70.- $\int \frac{xdx}{\sqrt{x^2+4x+5}}$

RESPUESTAS

6.17.- $\int \sqrt{4-x^2}$

Solución.-



Se tiene: $x = 2\operatorname{sen}\theta, dx = 2\cos\theta d\theta$, $\sqrt{4+x^2} = 2\cos\theta$

$$\int \sqrt{4-x^2} = \int 2\cos\theta 2\cos\theta d\theta = 4\int \cos^2\theta d\theta = 2\theta + \operatorname{sen}2\theta + c = 2\theta + 2\operatorname{sen}\theta\cos\theta + c$$

$$= 2\operatorname{arcsen}\frac{x}{2} + \frac{x\sqrt{4-x^2}}{2} + c$$

6.18.- $\int \frac{dx}{\sqrt{a^2-x^2}}$

Solución.- se tiene: $x = a\operatorname{sen}\theta, dx = a\cos\theta d\theta$, $\sqrt{a^2-x^2} = a\cos\theta$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{\cancel{a\cos\theta} d\theta}{\cancel{a\cos\theta}} = \int d\theta = \theta + c = \operatorname{arcsen}\frac{x}{a} + c$$

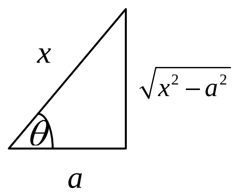
6.19.- $\int \frac{dx}{x^2+a^2}$

Solución.- se tiene: $x = a\tau g\theta, dx = a\sec^2\theta d\theta$, $\sqrt{x^2+a^2} = a\sec\theta$

$$\int \frac{dx}{x^2+a^2} = \int \frac{dx}{(\sqrt{x^2+a^2})^2} = \int \frac{\cancel{a}\,\cancel{\sec^2\theta} d\theta}{a^{\cancel{2}}\,\cancel{\sec^2\theta}} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + c = \frac{1}{a}\operatorname{arc}\tau g\frac{x}{a} + c$$

6.20.- $\int \frac{dx}{x^2-a^2}$

Solución.-



Se tiene: $x = a\sec\theta, dx = a\sec\theta\tau g\theta d\theta$, $\sqrt{x^2-a^2} = a\tau g\theta$

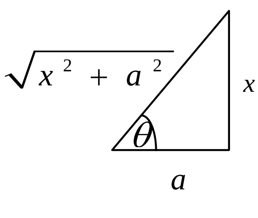
$$\int \frac{dx}{x^2-a^2} = \int \frac{dx}{(\sqrt{x^2-a^2})^2} = \int \frac{\cancel{a}\sec\theta\,\cancel{\tau g\theta} d\theta}{a^{\cancel{2}}\tau g^{\cancel{2}}\theta} = \frac{1}{a}\int \frac{\sec\theta d\theta}{\tau g\theta} = \frac{1}{a}\int \cos ec\theta d\theta$$

$$= \frac{1}{a}\ell\eta\left|\cos ec\theta - \operatorname{co}\tau g\theta\right| = \frac{1}{a}\ell\eta\left|\frac{x}{\sqrt{x^2-a^2}} - \frac{a}{\sqrt{x^2-a^2}}\right| + c$$

$$= \frac{1}{a}\ell\eta\left|\frac{x-a}{\sqrt{x^2-a^2}}\right| + c = \frac{1}{a}\ell\eta\left|\sqrt{\frac{(x-a)^2}{x^2-a^2}}\right| + c = \frac{1}{2a}\ell\eta\left|\frac{x-a}{x+a}\right| + c$$

6.21.- $\int \frac{dx}{\sqrt{x^2+a^2}}$

Solución.-



Se tiene: $x = a\tau g\theta, dx = a\sec^2\theta d\theta$, $\sqrt{x^2+a^2} = a\sec\theta$

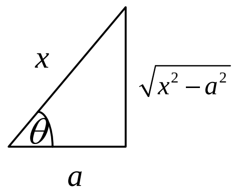
$$\int\frac{dx}{\sqrt{x^2+a^2}} = \int\frac{a\sec^2\theta d\theta}{a\sec\theta} = \int\sec\theta d\theta = \ell\eta\left|\sec\theta+\tau g\theta\right| + c$$

$$= \ell\eta\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right| + c = \ell\eta\left|\frac{\sqrt{x^2+a^2}+x}{a}\right| + c = \ell\eta\left|x+\sqrt{x^2+a^2}\right| - \ell\eta a + c$$

$$= \ell\eta\left|x+\sqrt{x^2+a^2}\right| + c$$

6.22.- $\int\frac{dx}{\sqrt{x^2-a^2}}$

Solución.-



Se tiene: $x = a\sec\theta, dx = a\sec\theta\tau g\theta d\theta$, $\sqrt{x^2+a^2} = a\tau g\theta$

$$\int\frac{dx}{\sqrt{x^2-a^2}} = \int\frac{a\sec\theta\,\tau g\theta d\theta}{a\tau g\theta} = \int\sec\theta d\theta = \ell\eta\left|\sec\theta+\tau g\theta\right| + c$$

$$= \ell\eta\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right| + c = \ell\eta\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + c = \ell\eta\left|x+\sqrt{x^2-a^2}\right| + c$$

6.23.- $\int\frac{dx}{x\sqrt{x^2-9}}$

Solución.-

Se tiene: $x = 3\sec\theta, dx = 3\sec\theta\tau g\theta d\theta$, $\sqrt{x^2-9} = 3\tau g\theta$

$$\int\frac{dx}{x\sqrt{x^2-9}} = \int\frac{3\sec\theta\,\tau g\theta d\theta}{3\sec\theta\, 3\tau g\theta} = \frac{1}{3}\int d\theta = \frac{1}{3}\theta + c = \frac{\operatorname{arcsec} x/3}{3} + c$$

6.24.- $\int\frac{dx}{x\sqrt{x^2-2}}$

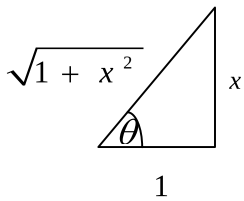
Solución.-

Se tiene: $x = \sqrt{2}\sec\theta, dx = \sqrt{2}\sec\theta\tau g\theta d\theta$, $\sqrt{x^2-2} = \sqrt{2}\tau g\theta$

$$\int\frac{dx}{x\sqrt{x^2-2}} = \int\frac{\sqrt{2}\sec\theta\,\tau g\theta d\theta}{\sqrt{2}\sec\theta\sqrt{2}\,\tau g\theta} = \frac{\sqrt{2}}{2}\int d\theta = \frac{\sqrt{2}}{2}\theta + c = \frac{\sqrt{2}}{2}\operatorname{arcsec}\frac{\sqrt{2}}{2}x + c$$

6.25.- $\int\frac{dx}{x\sqrt{1+x^2}}$

Solución.-



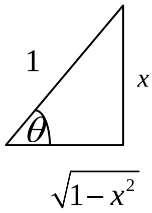
Se tiene: $x = \tau g\theta, dx = \sec^2\theta d\theta$, $\sqrt{1+x^2} = \sec\theta$

$$\int \frac{dx}{x\sqrt{1+x^2}} = \int \frac{\sec^{\cancel{2}}\theta d\theta}{\tau g\theta \cancel{\sec\theta}} = \int \frac{d\theta}{\mathrm{sen}\,\theta} = \int \cos ec\theta d\theta = \ell\eta\left|\cos ec\theta - \mathrm{co}\,\tau g\theta\right| + c$$

$$= \ell\eta\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + c = \ell\eta\left|\frac{\sqrt{1+x^2}-1}{x}\right| + c$$

6.26.- $\int \frac{x^2dx}{\sqrt{1-x^2}}$

Solución.-



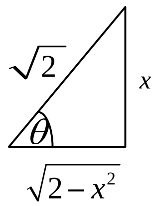
Se tiene: $x = \mathrm{sen}\,\theta, dx = \cos\theta d\theta$, $\sqrt{1-x^2} = \cos\theta$

$$\int \frac{x^2dx}{\sqrt{1-x^2}} = \int \frac{\mathrm{sen}^2\theta \cancel{\cos\theta} d\theta}{\cancel{\cos\theta}} = \int \mathrm{sen}^2\theta d\theta = \frac{1}{2}\theta - \frac{1}{4}\mathrm{sen}\,2\theta + c$$

$$= \frac{1}{2}\theta - \frac{1}{2}\mathrm{sen}\,\theta\cos\theta + c = \frac{\arcsin x}{2} - \frac{x}{2}\sqrt{1-x^2} + c$$

6.27.- $\int \frac{x^3dx}{\sqrt{2-x^2}}$

Solución.-



Se tiene: $x = \sqrt{2}\,\mathrm{sen}\,\theta, dx = \sqrt{2}\cos\theta d\theta, \sqrt{2-x^2} = \sqrt{2}\cos\theta$

$$\int \frac{x^3dx}{\sqrt{2-x^2}} = \int \frac{2\sqrt{2}\,\mathrm{sen}^3\theta \cancel{\sqrt{2}\cos\theta} d\theta}{\cancel{\sqrt{2}\cos\theta}} = 2\sqrt{2}\int \mathrm{sen}^3\theta d\theta = 2\sqrt{2}(-\cos\theta + \frac{\cos^3\theta}{3}) + c$$

$$= 2\sqrt{2}(-\frac{\sqrt{2-x^2}}{\sqrt{2}} + \frac{(\sqrt{2-x^2})^3}{3(\sqrt{2})^3}) + c = -\sqrt{2(2-x^2)} + \frac{(2-x^2)\sqrt{2-x^2}}{3} + c$$

6.28.- $\int \frac{\sqrt{x^2-9}}{x}dx$

Solución.-

Se tiene: $x = 3\sec\theta, dx = 3\sec\theta\tau g\theta d\theta$, $\sqrt{x^2-9} = 3\tau g\theta$

$$\int \frac{\sqrt{x^2-9}}{x}dx = \int \frac{3\tau g\theta \cancel{3\sec\theta}\tau g\theta d\theta}{\cancel{3\sec\theta}} = 3\int \tau g^2\theta d\theta = 3\int(\sec^2\theta - 1)d\theta$$

$$= 3\int \sec^2\theta d\theta - 3\int d\theta = 3\tau g\theta - 3\theta + c = \sqrt{x^2-9} - 3\,\mathrm{arc}\sec\frac{x}{3} + c$$

6.29.- $\int \frac{dx}{x\sqrt{4x^2-16}}$

Solución.-

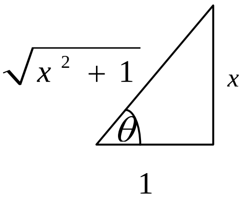
Se tiene: $\frac{x}{2}=\sec\theta, dx=2\sec\theta\tau g\theta d\theta$, $\sqrt{\frac{x^2}{4}-1}=\tau g\theta$

$$\int \frac{dx}{x\sqrt{4x^2-16}}=\frac{1}{4}\int \frac{dx}{x\sqrt{(x/2)^2-1}}=\frac{1}{4}\int \frac{2\sec\theta\tau g\theta d\theta}{2\sec\theta\tau g\theta}=\frac{1}{4}\int d\theta=\frac{1}{4}\theta+c$$

$$=\frac{1}{4}\operatorname{arc}\sec\frac{x}{2}+c$$

6.30.- $\int \frac{\sqrt{x^2+1}}{x}dx$

Solución.-



Se tiene: $x=\tau g\theta, dx=\sec^2\theta d\theta$, $\sqrt{x^2+1}=\sec\theta$

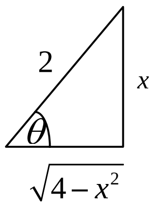
$$\int \frac{\sqrt{x^2+1}}{x}dx=\int \frac{\sec\theta\sec^2\theta d\theta}{\tau g\theta}=\int \frac{d\theta}{\cos^2\theta\, sen\,\theta}=\ell\eta\left|\tau g\frac{\theta}{2}\right|+\frac{1}{\cos\theta}+c$$, o bien:

$$=\ell\eta\left|\cos ec\theta-\operatorname{co}\tau g\theta\right|+\frac{1}{\cos\theta}+c=\ell\eta\left|\frac{\sqrt{x^2+1}}{x}-\frac{1}{x}\right|+\frac{1}{1/\sqrt{x^2+1}}+c$$

$$=\ell\eta\left|\frac{\sqrt{x^2+1}-1}{x}\right|+\sqrt{x^2+1}+c$$

6.31.- $\int \frac{dx}{x^2\sqrt{4-x^2}}$

Solución.-



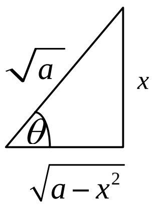
Se tiene: $x=2sen\,\theta, dx=2\cos\theta d\theta$, $\sqrt{4-x^2}=2\cos\theta$

$$\int \frac{dx}{x^2\sqrt{4-x^2}}=\int \frac{2\cos\theta d\theta}{4sen^2\theta\, 2\cos\theta}=\frac{1}{4}\int \cos ec^2\theta d\theta=-\frac{1}{4}\operatorname{co}\tau g\theta+c$$

$$=-\frac{\sqrt{4-x^2}}{4x}+c$$

6.32.- $\int \sqrt{a-x^2}dx$

Solución.-



Se tiene: $x = \sqrt{a}\,\mathrm{sen}\,\theta, dx = \sqrt{a}\cos\theta d\theta\,, \sqrt{a - x^2} = \sqrt{a}\cos\theta$

$$\int \sqrt{a - x^2}dx = \int \sqrt{a}\cos\theta\sqrt{a}\cos\theta d\theta = a\int \cos^2\theta d\theta$$

$$\frac{a}{2}\theta + \frac{a}{2}\mathrm{sen}\,\theta\cos\theta + c = \frac{a}{2}\mathrm{arcsen}\frac{x}{\sqrt{a}} + \frac{x}{2}\sqrt{a^2 - x^2} + c$$

6.33.- $\int \sqrt{a^2 - x^2}dx$

Solución.-

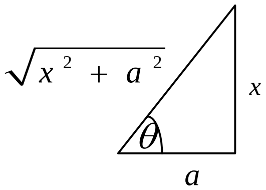
Se tiene: $x = a\,\mathrm{sen}\,\theta, dx = a\cos\theta d\theta\,, \sqrt{a^2 - x^2} = a\cos\theta$

$$\int \sqrt{a^2 - x^2}dx = \int a\cos\theta a\cos\theta d\theta = a^2\int \cos^2\theta d\theta$$

$$\frac{a^2}{2}\theta + \frac{a^2}{2}\mathrm{sen}\,\theta\cos\theta + c = \frac{a^2}{2}\mathrm{arcsen}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + c$$

6.34.- $\int \frac{x^2dx}{\sqrt{x^2 + a^2}}$

Solución.-



Se tiene: $x = a\tau g\theta, dx = a\sec^2\theta d\theta\,, \sqrt{x^2 + a^2} = a\sec\theta$

$$\int \frac{x^2dx}{\sqrt{x^2 + a^2}} = \int \frac{a^2\tau g^2\theta \cancel{a}\sec^{\cancel{2}}\theta d\theta}{\cancel{a\sec\theta}} = a^2\int \tau g^2\theta\sec\theta d\theta = a^2\int \frac{\mathrm{sen}^2\theta}{\cos^3\theta}d\theta$$

$$= a^2\int \frac{(1 - \cos^2\theta)}{\cos^3\theta}d\theta = a^2\int \sec^3\theta d\theta - a^2\int \sec\theta d\theta$$

$$= a^2\left(\frac{\sec\theta\tau g\theta}{2} + \frac{1}{2}\ell\eta\left|\sec\theta + \tau g\theta\right|\right) - a^2\ell\eta\left|\sec\theta + \tau g\theta\right| + c$$

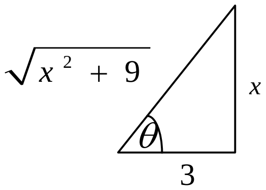
$$= \frac{a^2}{2}\sec\theta\tau g\theta + \frac{a^2}{2}\ell\eta\left|\sec\theta + \tau g\theta\right| - a^2\ell\eta\left|\sec\theta + \tau g\theta\right| + c$$

$$= \frac{a^2}{2}\sec\theta\tau g\theta - \frac{a^2}{2}\ell\eta\left|\sec\theta + \tau g\theta\right| + c$$

$$= \frac{a^{\cancel{2}}}{2}\frac{\sqrt{x^2 + a^2}}{\cancel{a}}\frac{x}{\cancel{a}} - \frac{a^2}{2}\ell\eta\left|\frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a}\right| + c = \frac{x\sqrt{x^2 + a^2}}{2} - \frac{a^2}{2}\ell\eta\left|\sqrt{x^2 + a^2} + x\right| + c$$

6.35.- $\int \frac{dx}{x^2\sqrt{x^2 + 9}}$

Solución.-



Se tiene: $x = 3\tau g\theta, dx = 3\sec^2\theta d\theta$, $\sqrt{x^2+9} = 3\sec\theta$

$$\int \frac{dx}{x^2\sqrt{x^2+9}} = \int \frac{3\sec^2\theta d\theta}{9\tau g^2\theta\, 3\sec\theta} = \frac{1}{9}\int \frac{\sec\theta d\theta}{\tau g^2\theta} = \frac{1}{9}\int \frac{\cos\theta}{\operatorname{sen}^2\theta} d\theta = -\frac{1}{9\operatorname{sen}\theta} + c$$

$$= -\frac{\sqrt{x^2+9}}{9x} + c$$

6.36.- $\int \frac{dx}{\sqrt{5-4x^2}}$

Solución.-

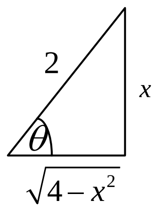
Se tiene: $x = \sqrt{5/4}\operatorname{sen}\theta, dx = \sqrt{5/4}\cos\theta d\theta$, $\sqrt{(5/4)^2 - x^2} = 5/4\cos\theta$

$$\int \frac{dx}{\sqrt{5-4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{5/4 - x^2}} = \frac{1}{2}\int \frac{\sqrt{5/4}\cos\theta d\theta}{\sqrt{5/4}\cos\theta} = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + c$$

$$= \frac{1}{2}\operatorname{arcsen}\frac{x}{\sqrt{5/4}} + c = \frac{1}{2}\operatorname{arcsen}\frac{2x}{\sqrt{5}} + c$$

6.37.- $\int \frac{x^2dx}{(4-x^2)^{3/2}}$

Solución.-



Se tiene: $x = 2\operatorname{sen}\theta, dx = 2\cos\theta d\theta$, $\sqrt{4-x^2} = 2\cos\theta$

$$\int \frac{x^2dx}{(4-x^2)^{3/2}} = \int \frac{x^2dx}{\sqrt{(4-x^2)^3}} = \int \frac{4\operatorname{sen}^2\theta\, 2\cos\theta d\theta}{8\cos^3\theta} = \int \tau g^2\theta d\theta = \int (\sec^2\theta - 1)d\theta$$

$$= \tau g\theta - \theta + c = \frac{x}{\sqrt{4-x^2}} - \operatorname{arcsen}\frac{x}{2} + c$$

6.38.- $\int x^2\sqrt{5-x^2}dx$

Solución.-

Se tiene: $x = \sqrt{5}\operatorname{sen}\theta, dx = \sqrt{5}\cos\theta d\theta$, $\sqrt{5-x^2} = \sqrt{5}\cos\theta$

$$\int x^2\sqrt{5-x^2}dx = \int 5\operatorname{sen}^2\theta\sqrt{5}\cos\theta\sqrt{5}\cos\theta d\theta = 25\int \operatorname{sen}^2\theta\cos^2\theta d\theta = \frac{25}{4}\int \operatorname{sen}^2 2\theta d\theta$$

$$= \frac{25}{8}\int (1-\cos 4\theta)d\theta = \frac{25}{8}\theta - \frac{25}{32}\operatorname{sen}4\theta + c = \frac{25}{8}\theta - \frac{25}{32}(2\operatorname{sen}2\theta\cos 2\theta) + c$$

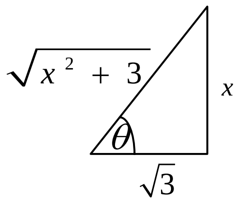
$$= \frac{25}{8}\theta - \frac{25}{32}\left[2\operatorname{sen}\theta\cos 2\theta(\cos^2\theta - \operatorname{sen}^2\theta)\right] + c$$

$$= \frac{25}{8}\theta - \frac{25}{16}\left[\operatorname{sen}\theta\cos^3\theta - \operatorname{sen}^3\theta\cos\theta) \right] + c$$

$$= \frac{25}{2}\left[\operatorname{arcsen}\frac{x}{\sqrt{5}} - \frac{x(\sqrt{5-x^2})^3}{25} + \frac{x^3\sqrt{5-x^2}}{25} \right] + c$$

6.39.- $\int \frac{dx}{x^4\sqrt{x^2+3}}$

Solución.-



Se tiene: $x = \sqrt{3}\tau g\theta, dx = \sqrt{3}\sec^2\theta d\theta$, $\sqrt{x^2+3} = \sqrt{3}\sec\theta$

$$\int \frac{dx}{x^4\sqrt{x^2+3}} = \int \frac{\sqrt{3}\sec^2\theta d\theta}{9\tau g^4\theta\sqrt{3}\sec\theta} = \frac{1}{9}\int \frac{\sec\theta d\theta}{\tau g^4\theta} = \frac{1}{9}\int \frac{\cos^3\theta d\theta}{\operatorname{sen}^4\theta} = \frac{1}{9}\int \frac{(1-\operatorname{sen}^2\theta)\cos\theta d\theta}{\operatorname{sen}^4\theta}$$

$$= \frac{1}{9}\int \frac{\cos\theta d\theta}{\operatorname{sen}^4\theta} - \frac{1}{9}\int \frac{\cos\theta d\theta}{\operatorname{sen}^2\theta} = -\frac{1}{27}\cos ec^3\theta + \frac{1}{9}\cos ec\theta + c = \frac{\sqrt{x^2+3}}{9x} - \left(\frac{\sqrt{x^2+3}}{3x} \right)^3 + c$$

6.40.- $\int x^3\sqrt{a^2x^2+b^2}dx$

Solución.-

Se tiene: $ax = b\tau g\theta, adx = b\sec^2\theta d\theta$, $\sqrt{a^2x^2+b^2} = b\sec\theta$

$$\int x^3\sqrt{a^2x^2+b^2}dx = \int \frac{b^3}{a^3}\tau g^3\theta b\sec\theta\frac{b}{a}\sec^2\theta d\theta = \frac{b^5}{a^4}\int \tau g^3\theta\sec^3\theta d\theta$$

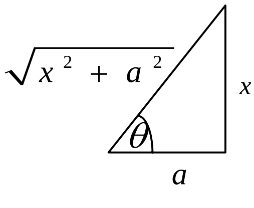
$$= \frac{b^5}{a^4}\int \tau g^2\theta\sec^2\theta\tau g\theta\sec\theta d\theta = \frac{b^5}{a^4}\int (\sec^2\theta - 1)\sec^2\theta\tau g\theta\sec\theta d\theta$$

$$= \frac{b^5}{a^4}\int \sec^4\theta\tau g\theta\sec\theta d\theta - \frac{b^5}{a^4}\int \sec^2\theta\tau g\theta\sec\theta d\theta = \frac{b^5}{a^4}\frac{\sec^5\theta}{5} + \frac{b^5}{a^4}\frac{\sec^3\theta}{3} + c$$

$$= \frac{b^5}{a^4}\left[\frac{(\sqrt{a^2x^2+b^2})^5}{5b^5} + \frac{(\sqrt{a^2x^2+b^2})^3}{3b^3} \right] + c = \frac{(a^2x^2+b^2)^{5/2}}{5a^4} - \frac{(a^2x^2+b^2)^{3/2}b^2}{3a^4} + c$$

6.41.- $\int \frac{dx}{x^2\sqrt{x^2+a^2}}$

Solución.-



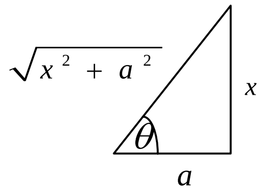
Se tiene: $x = a\tau g\theta, dx = a\sec^2\theta d\theta$, $\sqrt{x^2+a^2} = a\sec\theta$

$$\int \frac{dx}{x^2\sqrt{x^2+a^2}} = \int \frac{a\sec^2\theta d\theta}{a^2\tau g^2\theta\, a\sec\theta} = \frac{1}{a^2}\int \frac{\sec\theta d\theta}{\tau g^2\theta} = \frac{1}{a^2}\int \frac{\cos\theta d\theta}{\operatorname{sen}^2\theta} d\theta$$

$$= \frac{1}{a^2}\int \operatorname{co}\tau g\theta \operatorname{cosec}\theta d\theta = -\frac{\operatorname{cosec}\theta}{a^2} + c = -\frac{1}{a^2 x}\sqrt{x^2+a^2} + c$$

6.42.- $\int \frac{dx}{(x^2+a^2)^2}$

Solución.-



Se tiene: $x = a\tau g\theta, dx = a\sec^2\theta d\theta$, $\sqrt{x^2+a^2} = a\sec\theta$

$$\int \frac{dx}{(x^2+a^2)^2} = \int \frac{dx}{(\sqrt{x^2+a^2})^4} = \int \frac{a\sec^2\theta d\theta}{a^4\sec^4\theta} = \frac{1}{a^3}\int \cos^2\theta d\theta = \frac{1}{2a^3}\theta + \frac{1}{2a^3}\frac{\operatorname{sen}2\theta}{2} + c$$

$$= \frac{1}{2a^3}\theta + \frac{1}{2a^3}\frac{2\operatorname{sen}\theta\cos\theta}{2} + c = \frac{1}{2a^3}\operatorname{arc}\tau g\frac{x}{a} + \frac{1}{2a^3}\left(\frac{x}{\sqrt{x^2+a^2}}\frac{a}{\sqrt{x^2+a^2}}\right) + c$$

$$= \frac{1}{2a^3}\operatorname{arc}\tau g\frac{x}{a} + \frac{1}{2a^3}\left(\frac{ax}{\sqrt{x^2+a^2}}\right) + c$$

6.43.- $\int x^3\sqrt{a^2x^2-b^2}\,dx$

Solución.-

Se tiene: $ax = b\sec\theta, adx = b\sec\theta\tau g\theta d\theta$, $\sqrt{a^2x^2-b^2} = b\tau g\theta$

$$\int x^3\sqrt{a^2x^2-b^2}\,dx = \int \frac{b^3}{a^3}\sec^3\theta b\tau g\theta \frac{b}{a}\sec\theta\tau g\theta d\theta = \frac{b^5}{a^4}\int \sec^4\theta\tau g^2\theta d\theta$$

$$= \frac{b^5}{a^4}\int \sec^4\theta(\sec^2\theta - 1)d\theta = \frac{b^5}{a^4}\int \sec^4\theta\sec^2\theta d\theta - \frac{b^5}{a^4}\int \sec^2\theta\sec^2\theta d\theta$$

$$= \frac{b^5}{a^4}\int (1+\tau g^2\theta)^2\sec^2\theta d\theta - \frac{b^5}{a^4}\int (1+\tau g^2\theta)\sec^2\theta d\theta$$

$$= \frac{b^5}{a^4}\int (1+2\tau g^2\theta+\tau g^4\theta)\sec^2\theta d\theta - \frac{b^5}{a^4}\int (1+\tau g^2\theta)\sec^2\theta d\theta$$

$$= \frac{b^5}{a^4}\left[\int \tau g^2\theta\sec^2\theta d\theta + \int \tau g^4\theta\sec^2\theta d\theta\right] = \frac{b^5}{a^4}\left[\frac{\tau g^3\theta}{3} + \frac{\tau g^5\theta}{5}\right] + c$$

$$= \frac{b^5}{a^4}\left[\frac{1}{3}\left(\frac{\sqrt{a^2x^2-b^2}}{b}\right)^3 + \frac{1}{5}\left(\frac{\sqrt{a^2x^2-b^2}}{b}\right)^5\right] + c$$

6.44.- $\int \frac{dx}{x^2\sqrt{a^2-x^2}}$

Solución.-

Se tiene: $x = a\,\mathrm{sen}\,\theta, dx = a\cos\theta d\theta$, $\sqrt{a^2 - x^2} = a\cos\theta$

$$\int \frac{dx}{x^2\sqrt{a^2-x^2}} = \int \frac{\cancel{a\cos\theta}\, d\theta}{a^2\,\mathrm{sen}^2\,\theta\,\cancel{a\cos\theta}} = \frac{1}{a^2}\int \cos ec^2\theta d\theta = -\frac{1}{a^2}\mathrm{co}\,\tau g\theta + c$$

$$= -\frac{1}{a^2}\frac{\cos\theta}{\mathrm{sen}\,\theta} + c = -\frac{1}{a^2}\left(\frac{\sqrt{a^2-x^2}}{x}\right) + c$$

6.45.- $\int \frac{\sqrt{2x^2-5}}{x}dx$

Solución.-

Se tiene: $\sqrt{2}x = \sqrt{5}\sec\theta, \sqrt{2}dx = \sqrt{5}\sec\theta\tau g\theta d\theta$, $\sqrt{2x^2-5} = \sqrt{5}\tau g\theta$

$$\int \frac{\sqrt{2x^2-5}}{x}dx = \int \frac{\sqrt{5}\tau g\theta \cancel{\frac{\sqrt{5}}{\sqrt{2}}\sec\theta}\tau g\theta d\theta}{\cancel{\frac{\sqrt{5}}{\sqrt{2}}\sec\theta}} = \sqrt{5}\int \tau g^2\theta d\theta = \sqrt{5}\int \sec^2\theta d\theta - \sqrt{5}\int d\theta$$

$$= \sqrt{5}\tau g\theta - \sqrt{5}\theta + c = \sqrt{2x^2-5} - \sqrt{5}\,\mathrm{arc}\sec\sqrt{2/3}\,x + c$$

6.46.- $\int \frac{x^3dx}{\sqrt{3x^2-5}}$

Solución.-

Se tiene: $\sqrt{3}x = \sqrt{5}\sec\theta, \sqrt{3}dx = \sqrt{5}\sec\theta\tau g\theta d\theta$, $\sqrt{3x^2-5} = \sqrt{5}\tau g\theta$

$$\int \frac{x^3dx}{\sqrt{3x^2-5}} = \int \frac{(\sqrt{5/3}\sec\theta)^3\sqrt{5/3}\sec\theta\,\cancel{\tau g\theta}\,d\theta}{\sqrt{5/3}\,\cancel{\tau g\theta}} = \frac{5\sqrt{5}}{9}\int \sec^4\theta d\theta$$

$$= \frac{5\sqrt{5}}{9}\int \sec^2\theta\sec^2\theta d\theta = \frac{5\sqrt{5}}{9}\int \sec^2\theta(1+\tau g^2\theta)d\theta$$

$$= \frac{5\sqrt{5}}{9}\left[\int \sec^2\theta d\theta + \int \sec^2\theta\tau g^2\theta d\theta\right] = \frac{5\sqrt{5}}{9}\left[\tau g\theta + \frac{\tau g^3\theta}{3}\right] + c$$

$$= \frac{5}{9}\left[\sqrt{3x^2-5} + \frac{(\sqrt{3x^2-5})^3}{15}\right] + c$$

6.47.- $\int \frac{\sqrt{x^2-100}}{x}dx$

Solución.-

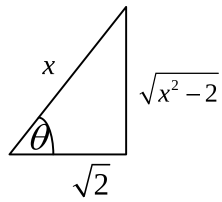
Se tiene: $x = 10\sec\theta, dx = 10\sec\theta\tau g\theta d\theta$, $\sqrt{x^2-100} = 10\tau g\theta$

$$\int \frac{\sqrt{x^2-100}}{x}dx = \int \frac{10\tau g\theta\,\cancel{10\sec\theta}\,\tau g\theta d\theta}{\cancel{10\sec\theta}} = 10\int \tau g^2\theta d\theta = 10\int \sec^2\theta - 10\int d\theta$$

$$= 10(\tau g\theta - \theta) + c = \sqrt{x^2-100} - 10\,\mathrm{arcsen}\frac{x}{10} + c$$

6.48.- $\int \frac{dx}{x^2\sqrt{x^2-2}}$

Solución.-



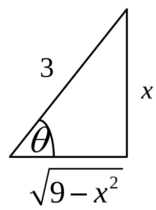
Se tiene: $x=\sqrt{2}\sec\theta, dx=\sqrt{2}\sec\theta\tau g\theta d\theta$, $\sqrt{x^2-2}=\sqrt{2}\tau g\theta$

$$\int \frac{dx}{x^2\sqrt{x^2-2}}=\int \frac{\sqrt{2}\sec\theta\;\tau g\theta d\theta}{2\sec^2\theta\sqrt{2}\tau g\theta}=\frac{1}{2}\int\cos\theta d\theta=\frac{1}{2}\operatorname{sen}\theta+c=\frac{1}{2}\frac{\sqrt{x^2-2}}{x}+c$$

$$=\frac{\sqrt{x^2-2}}{2x}+c$$

6.49.- $\int \frac{dx}{x\sqrt{9-x^2}}$

Solución.-



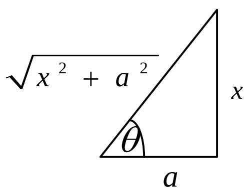
Se tiene: $x=3\operatorname{sen}\theta, dx=3\cos\theta d\theta$, $\sqrt{9-x^2}=3\cos\theta$

$$\int \frac{dx}{x\sqrt{9-x^2}}=\int \frac{3\cos\theta d\theta}{3\operatorname{sen}\theta\,3\cos\theta}=\frac{1}{3}\int\operatorname{cos}ec\theta d\theta=\frac{1}{3}\ell\eta\left|\operatorname{cos}ec\theta-\operatorname{co}\tau g\theta\right|+c$$

$$=\frac{1}{3}\ell\eta\left|\frac{3-\sqrt{9-x^2}}{x}\right|+c$$

6.50.- $\int \frac{\sqrt{x^2+a^2}}{x}dx$

Solución.-



Se tiene: $x=a\tau g\theta, dx=a\sec^2\theta d\theta$, $\sqrt{x^2+a^2}=a\sec\theta$

$$\int \frac{\sqrt{x^2+a^2}}{x}dx=\int \frac{a\sec\theta}{a\tau g\theta}a\sec^2\theta d\theta=a\int\frac{\sec^3\theta d\theta}{\tau g\theta}=a\int\frac{\sec^2\theta\sec\theta}{\tau g\theta}d\theta$$

$$=a\int\frac{(1+\tau g^2\theta)\sec\theta}{\tau g\theta}d\theta=a\int\frac{\sec\theta}{\tau g\theta}d\theta+a\int\sec\theta\tau g\theta d\theta$$

$$a\ell\eta\left|\operatorname{cos}ec\theta-\operatorname{co}\tau g\theta\right|+a\sec\theta+c=a\ell\eta\left|\frac{\sqrt{x^2+a^2}-a}{x}\right|+\sqrt{x^2+a^2}+c$$

6.51.- $\int \frac{xdx}{\sqrt{a^2-x^2}}$

Solución.-

Se tiene: $x = a\,\text{sen}\,\theta, dx = a\cos\theta d\theta$, $\sqrt{a^2-x^2} = a\cos\theta$

$$\int \frac{xdx}{\sqrt{a^2-x^2}} = \int \frac{a\,\text{sen}\,\theta\,\cancel{a\cos\theta}}{\cancel{a\cos\theta}}d\theta = a\int \text{sen}\,\theta d\theta = -a\cos\theta + c = -\sqrt{a^2-x^2} + c$$

6.52.- $\int \frac{dx}{\sqrt{1-4x^2}}$

Solución.-

Se tiene: $2x = \text{sen}\,\theta, 2dx = \cos\theta d\theta$, $\sqrt{1-4x^2} = \cos\theta$

$$\int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\int \frac{\cancel{\cos\theta}}{\cancel{\cos\theta}}d\theta = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + c = \frac{1}{2}\text{arcsen}\,2x + c$$

6.53.- $\int \frac{dx}{\sqrt{4+x^2}}$

Solución.-

Se tiene: $x = 2\tau g\theta, dx = 2\sec^2\theta d\theta$, $\sqrt{4+x^2} = 2\sec\theta$

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{\cancel{2}\sec^{\cancel{2}}\theta d\theta}{\cancel{2\sec\theta}} = \int \sec\theta d\theta = \ell\eta\left|\sec\theta + \tau g\theta\right| + c = \ell\eta\left|\sqrt{4+x^2} + x\right| + c$$

6.54.- $\int \frac{xdx}{\sqrt{4+x^2}}$

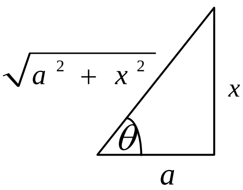
Solución.-

Se tiene: $x = 2\tau g\theta, dx = 2\sec^2\theta d\theta$, $\sqrt{4+x^2} = 2\sec\theta$

$$\int \frac{xdx}{\sqrt{4+x^2}} = \int \frac{2\tau g\theta\cancel{2}\sec^{\cancel{2}}\theta d\theta}{\cancel{2\sec\theta}} = 2\int \tau g\theta\sec\theta d\theta = 2\sec\theta + c = \sqrt{4+x^2} + c$$

6.55.- $\int \frac{dx}{x\sqrt{a^2+x^2}}$

Solución.-



Se tiene: $x = a\tau g\theta, dx = a\sec^2\theta d\theta$, $\sqrt{a^2+x^2} = a\sec\theta$

$$\int \frac{dx}{x\sqrt{a^2+x^2}} = \int \frac{\cancel{a}\sec^{\cancel{2}}\theta d\theta}{a\tau g\theta\,\cancel{a\sec\theta}} = \frac{1}{a}\int \frac{\sec\theta d\theta}{\tau g\theta} = \frac{1}{a}\int \text{cos}ec\theta d\theta$$

$$= \frac{1}{a}\ell\eta\left|\cos ec\theta - \text{co}\,\tau g\theta\right| + c = \frac{1}{a}\ell\eta\left|\frac{\sqrt{a^2+x^2}}{x} - \frac{a}{x}\right| + c = \frac{1}{a}\ell\eta\left|\frac{\sqrt{a^2+x^2} - a}{x}\right| + c$$

6.56.- $\int \frac{(x+1)dx}{\sqrt{4-x^2}}$

Solución.-

Se tiene: $x = 2\,\mathrm{sen}\,\theta, dx = 2\cos\theta d\theta$, $\sqrt{4-x^2} = 2\cos\theta$

$$\int \frac{(x+1)dx}{\sqrt{4-x^2}} = \int \frac{xdx}{\sqrt{4-x^2}} + \int \frac{dx}{\sqrt{4-x^2}} = \int \frac{2\,\mathrm{sen}\,\cancel{2\cos\theta}d\theta}{\cancel{2\cos\theta}} + \int \frac{\cancel{2\cos\theta}d\theta}{\cancel{2\cos\theta}}$$

$$2\int \mathrm{sen}\,\theta d\theta + \int d\theta = -2\cos\theta + \theta + c = -\sqrt{4-x^2} + \mathrm{arcsen}\frac{x}{2} + c$$

6.57.- $\int \frac{dx}{\sqrt{2-5x^2}}$

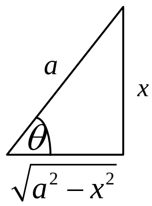
Solución.-

Se tiene: $\sqrt{5}x = \sqrt{2}\,\mathrm{sen}\,\theta, \sqrt{5}dx = \sqrt{2}\cos\theta d\theta$, $\sqrt{2-5x^2} = \sqrt{2}\cos\theta$

$$\int \frac{dx}{\sqrt{2-5x^2}} = \int \frac{\frac{\cancel{\sqrt{2}}}{\sqrt{5}}\cancel{\cos\theta}d\theta}{\cancel{\sqrt{2}}\cancel{\cos\theta}} = \frac{\sqrt{5}}{5}\int d\theta = \frac{\sqrt{5}}{5}\theta + c = \frac{\sqrt{5}}{5}\mathrm{arcsen}\sqrt{5/2}\,x + c$$

6.58.- $\int \frac{dx}{(a^2-x^2)^{3/2}}$

Solución.-



Se tiene: $x = a\,\mathrm{sen}\,\theta, dx = a\cos\theta d\theta$, $\sqrt{a^2-x^2} = a\cos\theta$

$$\int \frac{dx}{(a^2-x^2)^{3/2}} = \int \frac{dx}{(\sqrt{a^2-x^2})^3} = \int \frac{\cancel{a}\cancel{\cos\theta}d\theta}{a^{\cancel{3}}\cos^{\cancel{3}}\theta} = \frac{1}{a^2}\int \sec^2\theta d\theta = \frac{1}{a^2}\tau g\theta + c$$

$$= \frac{x}{a^2\sqrt{a^2-x^2}} + c$$

6.59.- $\int \frac{dx}{\sqrt{4-(x-1)^2}}$

Solución.-

Se tiene: $x-1 = 2\,\mathrm{sen}\,\theta, dx = 2\cos\theta d\theta$, $\sqrt{4-(x-1)^2} = 2\cos\theta$

$$\int \frac{dx}{\sqrt{4-(x-1)^2}} = \int \frac{\cancel{2\cos\theta}d\theta}{\cancel{2\cos\theta}} = \int d\theta = \theta + c = \mathrm{arcsen}\frac{x-1}{2} + c$$

6.60.- $\int \frac{x^2dx}{\sqrt{2x-x^2}}$

Solución.-

Se tiene: $x-1 = \mathrm{sen}\,\theta \Rightarrow x = \mathrm{sen}\,\theta + 1, dx = \cos\theta d\theta$, $\sqrt{1-(x-1)^2} = \cos\theta$

Completando cuadrados se tiene:

$2x-x^2 = -(x^2-2x) = -(x^2-2x+1)+1 = 1-(x-1)^2$, luego:

$$\int \frac{x^2dx}{\sqrt{2x-x^2}} = \int \frac{x^2dx}{\sqrt{1-(x-1)^2}} = \int \frac{(\mathrm{sen}\,\theta+1)^2\cancel{\cos\theta}d\theta}{\cancel{\cos\theta}} = \int (\mathrm{sen}\,\theta+1)^2 d\theta$$

$= \int \operatorname{sen}^2\theta d\theta + 2\int \operatorname{sen}\theta d\theta + \int d\theta = \frac{1}{2}\int d\theta - \frac{1}{2}\int \cos 2\theta d\theta + 2\int \operatorname{sen}\theta d\theta + \int d\theta$

$= \frac{3}{2}\int d\theta - \frac{1}{2}\int \cos 2\theta d\theta + 2\int \operatorname{sen}\theta d\theta = \frac{3}{2}\theta - \frac{1}{4}\operatorname{sen} 2\theta - 2\cos\theta + c$

$= \frac{3}{2}\theta - \frac{1}{2}\operatorname{sen}\theta\cos\theta - 2\cos\theta + c = \frac{3}{2}\operatorname{arcsen}(x-1) - \frac{1}{2}(x-1)\sqrt{2x-x^2} - 2\sqrt{2x-x^2} + c$

6.61.- $\int \frac{x^2dx}{\sqrt{17-x^2}}$

Solución.-

Se tiene: $x = \sqrt{17}\operatorname{sen}\theta, dx = \sqrt{17}\cos\theta d\theta$, $\sqrt{17-x^2} = \sqrt{17}\cos\theta$

$\int \frac{x^2dx}{\sqrt{17-x^2}} = \int \frac{17\operatorname{sen}^2\theta \cancel{\sqrt{17}\cos\theta} d\theta}{\cancel{\sqrt{17}\cos\theta}} = 17\int \operatorname{sen}^2\theta d\theta = \frac{17}{2}\int d\theta - \frac{17}{2}\int \cos 2\theta d\theta$

$= \frac{17}{2}\theta - \frac{17}{4}\operatorname{sen} 2\theta + c = \frac{17}{2}\theta - \frac{17}{2}\operatorname{sen}\theta\cos\theta + c$

$= \frac{17}{2}\operatorname{arcsen}\frac{x}{\sqrt{17}} - \frac{\cancel{17}}{2}\frac{x}{\cancel{\sqrt{17}}}\frac{\sqrt{17-x^2}}{\cancel{\sqrt{17}}} + c = \frac{17}{2}\operatorname{arcsen}\frac{x}{\sqrt{17}} - \frac{1}{2}x\sqrt{17-x^2} + c$

6.62.- $\int \frac{x^2dx}{\sqrt{21+4x-x^2}}$

Solución.-

Se tiene: $x-2 = 5\operatorname{sen}\theta \Rightarrow x = 5\operatorname{sen}\theta + 2, dx = 5\cos\theta d\theta$, $\sqrt{5^2-(x-2)^2} = 5\cos\theta$

Completando cuadrados se tiene:

$21+4x-x^2 = -(x^2-4x+4-4)+21 = -(x^2-4x+4)+25 = 5^2-(x-2)^2$, luego:

$\int \frac{x^2dx}{\sqrt{21+4x-x^2}} = \int \frac{x^2dx}{\sqrt{5^2-(x-2)^2}} = \int \frac{(5\operatorname{sen}\theta+2)^2\cancel{5\cos\theta}d\theta}{\cancel{5\cos\theta}} = \int (5\operatorname{sen}\theta+2)^2 d\theta$

$= \int (25\operatorname{sen}^2\theta + 20\operatorname{sen}\theta + 4)d\theta = 25\int \frac{1-\cos 2\theta}{2}d\theta + 20\int \operatorname{sen}\theta d\theta + 4\int d\theta$

$= \frac{25}{2}\int d\theta - \frac{25}{2}\int \cos 2\theta d\theta + 20\int \operatorname{sen}\theta d\theta = \frac{25}{2}\theta - \frac{25}{4}\operatorname{sen} 2\theta - 20\cos\theta + 4\theta + c$

$= \frac{33}{2}\theta - \frac{25}{2}\operatorname{sen}\theta\cos\theta - 20\cos\theta + c$

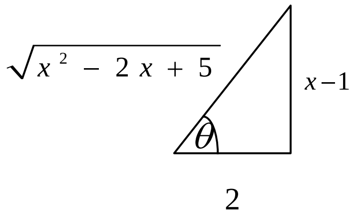
$= \frac{33}{2}\operatorname{arcsen}\frac{x-2}{5} - \frac{25}{2}\frac{x-2}{5}\left(\frac{\sqrt{21+4x-x^2}}{5}\right) - 20\left(\frac{\sqrt{21+4x-x^2}}{5}\right) + c$

$= \frac{33}{2}\operatorname{arcsen}\frac{x-2}{5} - \sqrt{21+4x-x^2}(\frac{x-2}{2}+4) + c$

$= \frac{33}{2}\operatorname{arcsen}\frac{x-2}{5} - \sqrt{21+4x-x^2}(\frac{x+6}{2}) + c$

6.63.- $\int \frac{dx}{(x^2-2x+5)^{3/2}}$

Solución.-



Se tiene: $x-1=2\tau g\theta, dx=2\sec^2\theta d\theta$, $\sqrt{(x-1)^2+2^2}=2\sec\theta$

Completando cuadrados se tiene:

$x^2-2x+5=(x^2-2x+1)+5-1=(x^2-2x+1)+4=(x-1)^2+2^2$, luego:

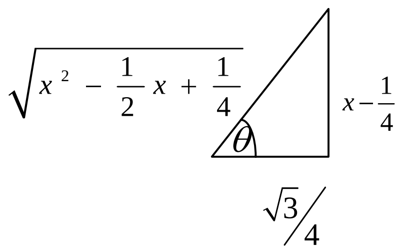
$$\int \frac{dx}{(x^2-2x+5)^{3/2}}=\int \frac{dx}{\sqrt{\left[(x-1)^2+2^2\right]^3}}=\int \frac{2\sec^2\theta d\theta}{2^3\sec^3\theta}=\frac{1}{4}\int \cos\theta d\theta=\frac{1}{4}\operatorname{sen}\theta+c$$

$$=\frac{1}{4}\frac{x-1}{\sqrt{x^2-2x+5}}+c$$

6.64.- $\int \frac{(2x+1)dx}{\sqrt{(4x^2-2x+1)^3}}$

Solución.-

Sea: $u=4x^2-2x+1, du=(8x-2)dx$



Se tiene: $x-\frac{1}{4}=\frac{\sqrt{3}}{4}\tau g\theta, dx=\frac{\sqrt{3}}{4}\sec^2\theta d\theta$, $\sqrt{(x-\frac{1}{4})^2+(\frac{\sqrt{3}}{4})^2}=\frac{\sqrt{3}}{4}\sec\theta$

Completando cuadrados se tiene:

$x^2-\frac{1}{2}x+\frac{1}{4}=(x^2-\frac{1}{2}x+\frac{1}{16})+\frac{1}{4}-\frac{1}{16}=(x-\frac{1}{4})^2+\frac{3}{16}=(x-\frac{1}{4})^2+(\frac{\sqrt{3}}{4})^2$, luego:

$$\int \frac{(2x+1)dx}{\sqrt{(4x^2-2x+1)^3}}=\frac{1}{4}\int \frac{(8x+4)dx}{\sqrt{(4x^2-2x+1)^3}}=\frac{1}{4}\int \frac{(8x-2+6)dx}{\sqrt{(4x^2-2x+1)^3}}$$

$$=\frac{1}{4}\int \frac{(8x-2)dx}{\sqrt{(4x^2-2x+1)^3}}+\frac{3}{2}\int \frac{dx}{\sqrt{(4x^2-2x+1)^3}}$$

$$=\frac{1}{4}\int \frac{du}{(u)^{3/2}}+\frac{3}{2}\int \frac{dx}{\sqrt{4(x^2-\frac{1}{2}x+\frac{1}{4})^3}}=\frac{1}{4}\int (u)^{-3/2}du+\frac{3}{2}\frac{1}{8}\int \frac{dx}{\sqrt{(x^2-\frac{1}{2}x+\frac{1}{4})^3}}$$

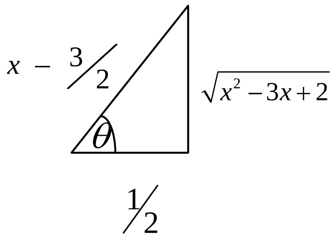
$$=\frac{1}{4}\int (u)^{-3/2}du+\frac{3}{16}\int \frac{dx}{\sqrt{\left[(x-\frac{1}{4})^2+(\frac{\sqrt{3}}{4})^2\right]^3}}=\frac{1}{4}\int (u)^{-3/2}du+\frac{3}{16}\int \frac{\frac{\sqrt{3}}{4}\sec^2\theta d\theta}{(\frac{\sqrt{3}}{4}\sec\theta)^3}$$

$$= \frac{1}{4}\int (u)^{-3/2}du + \int \frac{d\theta}{\sec\theta} = \frac{1}{4}\frac{u^{-1/2}}{(-1/2)} + \operatorname{sen}\theta + c = -\frac{1}{2u^{1/2}} + \operatorname{sen}\theta + c$$

$$= \frac{-1}{2\sqrt{4x^2-2x+1}} + \frac{x-1/4}{\sqrt{x^2-1/2\,x+1/4}} + c = \frac{4x-2}{4\sqrt{x^2-1/2\,x+1/4}} + c$$

6.65.- $\int \frac{dx}{(x-1)\sqrt{x^2-3x+2}}$

Solución.-



Se tiene: $x - \frac{3}{2} = \frac{1}{2}\sec\theta \Rightarrow x - 1 = \frac{1}{2}(\sec\theta + 1), dx = \frac{1}{2}\sec\theta\tau g\theta d\theta$,

$\sqrt{(x-3/2)^2 + (1/2)^2} = 1/2\,\tau g\theta$

Completando cuadrados se tiene:

$x^2 - 3x + 2 = (x^2 - 3x + \frac{9}{4}) - \frac{1}{4} = (x - \frac{3}{2})^2 - (\frac{1}{2})^2$, luego:

$$\int \frac{dx}{(x-1)\sqrt{x^2-3x+2}} = \int \frac{dx}{(x-1)\sqrt{(x-\frac{3}{2})^2-(\frac{1}{2})^2}} = \int \frac{1/2\sec\theta\,\tau g\theta\,d\theta}{1/2(\sec\theta+1)\,1/2\,\tau g\theta}$$

$$= \int \frac{\sec\theta d\theta}{1/2(\sec\theta+1)} = 2\int \frac{\sec\theta d\theta}{(\sec\theta+1)} = 2\int \frac{\sec\theta(\sec\theta-1)d\theta}{\sec^2\theta-1} = 2\int \frac{\sec^2\theta d\theta}{\tau g^2\theta} - 2\int \frac{\sec\theta d\theta}{\tau g^2\theta}$$

$$= 2\int \cos ec^2\theta d\theta - 2\int \frac{\operatorname{cosec}\theta d\theta}{\operatorname{sen}^2\theta} = -2\operatorname{co}\tau g\theta + 2\operatorname{cosec}\theta + c$$

$$-2\frac{1/2}{\sqrt{x^2-3x+2}} + 2\frac{x-3/2}{\sqrt{x^2-3x+2}} + c = \frac{2x-4}{\sqrt{x^2-3x+2}} + c$$

6.66.- $\int \frac{xdx}{\sqrt{x^2-2x+5}}$

Solución.-

Se tiene: $x - 1 = 2\tau g\theta, dx = 2\sec^2\theta d\theta$, $\sqrt{(x-1)^2 + (2)^2} = 2\sec\theta$

Completando cuadrados se tiene:

$x^2 - 2x + 5 = (x^2 - 2x + 1) + 4 = (x-1)^2 - 2^2$, luego:

$$\int\frac{xdx}{\sqrt{x^2-2x+5}}=\int\frac{xdx}{\sqrt{(x-1)^2-2^2}}=\int\frac{(2\tau g\theta+1)2\sec^2\theta d\theta}{2\sec\theta}$$

$$=2\int\tau g\theta\sec\theta d\theta+\int\sec\theta d\theta=2\sec\theta+\ell\eta\left|\sec\theta+\tau g\theta\right|+c$$

$$=\sqrt{x^2-2x+5}+\ell\eta\left|\frac{\sqrt{x^2-2x+5}+x-1}{2}\right|+c$$

6.67.- $\int\frac{(x+1)dx}{\sqrt{2x-x^2}}$

Solución.-

Se tiene: $x-1=\operatorname{sen}\theta\Rightarrow x+1=\operatorname{sen}\theta+2, dx=\cos\theta d\theta$, $\sqrt{1-(x-1)^2}=\cos\theta$

Completando cuadrados se tiene:

$2x-x^2=-(x^2-2x)=-(x^2-2x+1-1)=-(x^2-2x+1)+1=1-(x-1)^2$, luego:

$$\int\frac{(x+1)dx}{\sqrt{2x-x^2}}=\int\frac{(x+1)dx}{\sqrt{1-(x-1)^2}}=\int\frac{(\operatorname{sen}\theta+2)\cos\theta d\theta}{\cos\theta}=\int\operatorname{sen}\theta d\theta+2\int d\theta$$

$$=-\cos\theta+2\theta+c=-\sqrt{2x-x^2}+2\operatorname{arcsen}(x-1)+c$$

6.68.- $\int\frac{(x-1)dx}{\sqrt{x^2-4x+3}}$

Solución.-

Se tiene: $x-2=\sec\theta\Rightarrow x-1=\sec\theta+1, dx=\sec\theta\tau g\theta d\theta$, $\sqrt{(x-2)^2-1}=\tau g\theta$

Completando cuadrados se tiene:

$x^2-4x+3=x^2-4x+4-1=(x-2)^2-1$, luego:

$$\int\frac{(x-1)dx}{\sqrt{x^2-4x+3}}=\int\frac{(x-1)dx}{\sqrt{(x-2)^2-1}}=\int\frac{(\sec\theta+1)\sec\theta\,\tau g\theta d\theta}{\tau g\theta}$$

$$=\int\sec^2\theta d\theta+\int\sec\theta d\theta=\tau g\theta+\ell\eta\left|\sec\theta+\tau g\theta\right|+c$$

$$=\sqrt{x^2-4x+3}+\ell\eta\left|x-2+\sqrt{x^2-4x+3}\right|+c$$

6.69.- $\int\frac{dx}{\sqrt{x^2-2x-8}}$

Solución.-

Se tiene: $x-1=3\sec\theta, dx=3\sec\theta\tau g\theta d\theta$, $\sqrt{(x-1)^2-3^2}=3\tau g\theta$

Completando cuadrados se tiene:

$x^2-2x-8=x^2-2x+1-9=(x-1)^2-3^2$, luego:

$$\int\frac{dx}{\sqrt{x^2-2x-8}}=\int\frac{dx}{\sqrt{(x-1)^2-3^2}}=\int\frac{3\sec\theta\,\tau g\theta d\theta}{3\tau g\theta}=\int\sec\theta d\theta=\ell\eta\left|\sec\theta+\tau g\theta\right|+c$$

$$=\ell\eta\left|\frac{x-1}{3}+\frac{\sqrt{x^2-2x-8}}{3}\right|+c=\ell\eta\left|x-1+\sqrt{x^2-2x-8}\right|+c$$

6.70.- $\int \frac{xdx}{\sqrt{x^2+4x+5}}$

Solución.-

Se tiene: $x+2=\tau g\theta, dx=\sec^2\theta d\theta$, $\sqrt{(x+2)^2+1^2}=\sec\theta$

Completando cuadrados se tiene:

$x^2+4x+5=(x^2+4x+4)+1=(x+2)^2+1^2$, luego:

$$\int \frac{xdx}{\sqrt{x^2+4x+5}}=\int \frac{xdx}{\sqrt{(x+2)^2+1^2}}=\int \frac{(\tau g\theta-2)\sec^{\not{2}}\theta d\theta}{\not{\sec\theta}}=\int \tau g\theta\sec\theta d\theta-2\int \sec\theta d\theta$$

$$=\sec\theta-2\ell\eta\left|\sec\theta+\tau g\theta\right|+c=\sqrt{x^2+4x+5}-2\ell\eta\left|\sqrt{x^2+4x+5}+x+2\right|+c$$