Resuelve las siguientes integrales indefinidas de funciones racionales:

$$a)\ \int \frac{x^2+1}{x^3+x^2-2x}dx$$

$$b)\ \int \frac{x^2+1}{x^3-6x^2+12x-8}dx$$

$$c)\ \int \frac{4}{x^3-5x^2+7x-3}dx$$

$$d)\ \int \frac{x^2-3x+2}{x^2-9}dx$$

$$e)\ \int \frac{x^2+x-8}{x^3-4x^2}dx$$

$$f)\ \int \frac{2x^2-x+1}{x^2+8x+16}dx$$

$$g)\ \int \frac{x^4+3x^3-2x^2+1}{x+5}dx$$

Solución

$$a)\ \int \frac{x^2+1}{x^3+x^2-2x}dx \Rightarrow x^3+x^2-2x = x\left(x^2+x-2\right) = x\left(x-1\right)\left(x+2\right)$$

$$\frac{x^2+1}{x^3+x^2-2x} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+2}$$

$$x^2+1 = A\left(x-1\right)\left(x+2\right)+Bx\left(x+2\right)+Cx\left(x-1\right)$$

$$\left.\begin{array}{llll} x=0: & 1 & = & -2A \\ x=1: & 2 & = & 3B \\ x=-2: & 5 & = & 6C \end{array}\right\} \Rightarrow \begin{array}{l} A=-1/2 \\ B=2/3 \\ C=5/6 \end{array}$$

$$\int \frac{x^2+1}{x^3+x^2-2x}dx = -\frac{1}{2}\int\frac{dx}{x}+\frac{2}{3}\int\frac{dx}{x-1}+\frac{5}{6}\int\frac{dx}{x+2} =$$

$$= -\frac{1}{2}\ln|x|+\frac{2}{3}\ln|x-1|+\frac{5}{6}\ln|x+2|+C$$

$$b)\ \int \frac{x^2+1}{x^3-6x^2+12x-8}dx \Rightarrow x^3-6x^2+12x-8 = \left(x-2\right)^3$$

$$\frac{x^2+1}{\left(x-2\right)^3} = \frac{A}{x-2}+\frac{B}{\left(x-2\right)^2}+\frac{C}{\left(x-2\right)^3}$$

$$x^2+1 = A\left(x-2\right)^2+B\left(x-2\right)+C = Ax^2-4Ax+4A+Bx-2B+C$$

$$\left.\begin{array}{llll} x^2: & 1 & = & A \\ x^1: & 0 & = & -4A+B \\ x^0: & 1 & = & 4A-2B+C \end{array}\right\} \Rightarrow \begin{array}{l} A=1 \\ B=4 \\ C=5 \end{array}$$

$$\int \frac{x^2+1}{x^3-6x^2+12x-8}dx = \int \frac{dx}{x-2} + 4\int \frac{dx}{(x-2)^2} + 5\int \frac{dx}{(x-2)^3} =$$

$$= \ln|x-2| - \frac{4}{x-2} - \frac{5}{2(x-2)^2} + C$$

c) $\int \frac{4}{x^3-5x^2+7x-3}dx \Rightarrow x^3-5x^2+7x-3=(x-1)^2(x-3)$

$$\frac{4}{(x-1)^2(x-3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-3}$$

$$4 = A(x-1)(x-3) + B(x-3) + C(x-1)^2$$

$$\left.\begin{array}{llll} x=1: & 4 & = & -2B \\ x=3: & 4 & = & 4C \\ x=0: & 4 & = & 3A-3B+C \end{array}\right\} \Rightarrow \begin{array}{l} A=-1 \\ B=-2 \\ C=1 \end{array}$$

$$\int \frac{4}{x^3-5x^2+7x-3}dx = -\int \frac{dx}{x-1} - 2\int \frac{dx}{(x-1)^2} + \int \frac{dx}{x-3} =$$

$$= -\ln|x-1| + \frac{2}{x-1} + \ln|x-3| + C$$

d) $\int \frac{x^2-3x+2}{x^2-9}dx \Rightarrow x^2-3x+2 = 1\cdot(x^2-9) + (-3x+11)$

$$\int \frac{x^2-3x+2}{x^2-9}dx = \int dx + \int \frac{-3x+11}{x^2-9}dx \Rightarrow x^2-9=(x-3)(x+3)$$

$$\frac{-3x+11}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3} \Rightarrow -3x+11 = A(x+3) + B(x-3)$$

$$\left.\begin{array}{llll} x=\ \ 3: & 2 & = & 6A \\ x=-3: & 20 & = & -6B \end{array}\right\} \Rightarrow \begin{array}{l} A=1/3 \\ B=-10/3 \end{array}$$

$$\int \frac{x^2-3x+2}{x^2-9}dx = \int dx + \int \frac{-3x+11}{x^2-9}dx = x + \frac{1}{3}\int \frac{dx}{x-3} - \frac{10}{3}\int \frac{dx}{x+3} =$$

$$= x + \frac{1}{3}\ln|x-3| - \frac{10}{3}\ln|x+3| + C$$

e) $\int \frac{x^2+x-8}{x^3-4x^2}dx \Rightarrow x^3-4x^2=x^2(x-4)$

$$\frac{x^2+x-8}{x^3-4x^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-4}$$

$$x^2+x-8=Ax(x-4)+B(x-4)+Cx^2=Ax^2-4Ax+Bx-4B+Cx^2$$

$$\left.\begin{array}{lccc} x^2: & 1 & = & A+C \\ x^1: & 1 & = & -4A+B \\ x^0: & -8 & = & -4B \end{array}\right\} \Rightarrow \begin{array}{c} A=1/4 \\ B=2 \\ C=3/4 \end{array}$$

$$\int \frac{x^2+x-8}{x^3-4x^2}dx=\frac{1}{4}\int\frac{1}{x}dx+2\int\frac{1}{x^2}dx+\frac{3}{4}\int\frac{1}{(x-4)}dx=$$

$$=\frac{1}{4}\ln|x|-\frac{2}{x}+\frac{3}{4}\ln|x-4|+C$$

f) $\int \frac{2x^2-x+1}{x^2+8x+16}dx \Rightarrow 2x^2-x+1=2\cdot(x^2+8x+16)+(-17x-31)$

$$\int \frac{2x^2-x+1}{x^2+8x+16}dx=2\int dx+\int\frac{-17x-31}{x^2+8x+16}dx \Rightarrow x^2+8x+16=(x+4)^2$$

$$\frac{-17x-31}{(x+4)^2}=\frac{A}{x+4}+\frac{B}{(x+4)^2} \Rightarrow -17x-31=A(x+4)+B$$

$$\left.\begin{array}{lccc} x^1: & -17 & = & A \\ x^0: & -31 & = & 4A+B \end{array}\right\} \Rightarrow \begin{array}{c} A=-17 \\ B=37 \end{array}$$

$$\int \frac{2x^2-x+1}{x^2+8x+16}dx=2\int dx-17\int\frac{1}{x+4}dx+37\int\frac{1}{(x+4)^2}dx=$$

$$=2x-17\ln|x+4|-\frac{37}{x+4}+C$$

g) $\int \frac{x^4+3x^3-2x^2+1}{x+5}dx$

$$x^4+3x^3-2x^2+1=(x^3-2x^2+8x-40)\cdot(x+5)+201$$

$$\int \frac{x^4+3x^3-2x^2+1}{x+5}dx=\int(x^3-2x^2+8x-40)dx+\int\frac{201}{x+5}dx=$$

$$=\frac{x^4}{4}-2\frac{x^3}{3}+8\frac{x^2}{2}-40x+201\ln|x+5|+C$$