

## **INTEGRACION DE FUNCIONES IRRACIONALES**

En el caso de que el integrando contiene potencias fraccionarias de la variable de integración, estas se simplifican usando una sustitución del tipo:

$x = t^n$ ,  $\sqrt[n]{x} = t$ , siendo "n" el m.c.m de los denominadores de los exponentes.

### **EJERCICIOS DESARROLLADOS**

**9.1.-Encontrar:**  $\int \frac{\sqrt{x}dx}{1+x}$

Solución.- La única expresión "irracional" es  $\sqrt{x}$ , por lo tanto:

$\sqrt{x} = t \Rightarrow x = t^2, dx = 2tdt$ , luego:

$$\int \frac{\sqrt{x}dx}{1+x} = \int \frac{t(2tdt)}{1+t^2} = 2 \int \frac{t^2 dt}{1+t^2} = 2 \int \left(1 - \frac{1}{1+t^2}\right) dt = 2 \int dt - 2 \int \frac{dt}{t^2+1} = 2t - 2 \operatorname{arc} \tau g t + c$$

Dado que:  $t = \sqrt{x}$ , se tiene:  $= 2\sqrt{x} - 2 \operatorname{arc} \tau g \sqrt{x} + c$

**Respuesta:**  $\int \frac{\sqrt{x}dx}{1+x} = 2\sqrt{x} - 2 \operatorname{arc} \tau g \sqrt{x} + c$

**9.2.-Encontrar:**  $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})}$

Solución.- Análogamente al caso anterior:  $\sqrt{x} = t \Rightarrow x = t^2, dx = 2tdt$ , luego:

$$\int \frac{dx}{\sqrt{x}(1+\sqrt{x})} = \int \frac{2tdt}{t(1+t)} = \int \frac{2dt}{1+t} = 2 \operatorname{arc} \eta |t+1| + c$$

Dado que:  $t = \sqrt{x}$ , se tiene:  $= 2 \operatorname{arc} \eta |\sqrt{x}+1| + c$

**Respuesta:**  $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})} = 2 \operatorname{arc} \eta |\sqrt{x}+1| + c$

**9.3.-Encontrar:**  $\int \frac{dx}{3+\sqrt{x+2}}$

Solución.- La expresión "irracional" es ahora  $\sqrt{x+2}$ , por lo tanto:

$\sqrt{x+2} = t \Rightarrow x = t^2 - 2, dx = 2tdt$ , luego:

$$\int \frac{dx}{3+\sqrt{x+2}} = \int \frac{2tdt}{3+t} = 2 \int \left(1 - \frac{3}{t+3}\right) dt = 2 \int dt - 6 \int \frac{dt}{t+3} = 2t - 6 \operatorname{arc} \eta |t+3| + c$$

Dado que:  $t = \sqrt{x+2}$ , se tiene:  $= 2\sqrt{x+2} - 6 \operatorname{arc} \eta |\sqrt{x+2}+3| + c$

**Respuesta:**  $\int \frac{dx}{3+\sqrt{x+2}} = 2\sqrt{x+2} - 6 \operatorname{arc} \eta |\sqrt{x+2}+3| + c$

**9.4.-Encontrar:**  $\int \frac{1-\sqrt{3x+2}}{1+\sqrt{3x+2}} dx$

Solución.- La expresión “irracional” es ahora  $\sqrt{3x+2}$ , por lo tanto:

$$\sqrt{3x+2} = t \Rightarrow 3x = t^2 - 2, dx = \frac{2}{3}tdt, \text{ luego:}$$

$$\int \frac{1-\sqrt{3x+2}}{1+\sqrt{3x+2}} dx = \int \frac{1-t}{1+t} \cdot \frac{2}{3}tdt = \frac{2}{3} \int \frac{t-t^2}{1+t} dt = \frac{2}{3} \int \left( -t + 2 - \frac{2}{t+1} \right) dt$$

$$= -\frac{2}{3} \int tdt + \frac{4}{3} \int dt - \frac{4}{3} \int \frac{dt}{t+1} = -\frac{1}{3}t^2 + \frac{4}{3}t - \frac{4}{3}\ell\eta|t+1| + c$$

Dado que:  $t = \sqrt{3x+2}$ , se tiene:

$$= -\frac{1}{3}(3x+2) + \frac{4}{3}\sqrt{3x+2} - \frac{4}{3}\ell\eta|\sqrt{3x+2}+1| + c$$

$$= -x - \frac{2}{3} + \frac{4}{3}\sqrt{3x+2} - \frac{4}{3}\ell\eta|\sqrt{3x+2}+1| + c = -x - \frac{2}{3} + \frac{4}{3}(\sqrt{3x+2} - \ell\eta|\sqrt{3x+2}+1|) + c$$

**Respuesta:**  $\int \frac{1-\sqrt{3x+2}}{1+\sqrt{3x+2}} dx = -x - \frac{2}{3} + \frac{4}{3}(\sqrt{3x+2} - \ell\eta|\sqrt{3x+2}+1|) + c$

**9.5.- Encontrar:**  $\int \sqrt{1+\sqrt{x}} dx$

Solución.- La expresión “irracional” es ahora  $\sqrt{x}$ , por lo tanto:

$\sqrt{x} = t \Rightarrow x = t^2, dx = 2tdt$ , luego:  $\int (\sqrt{1+\sqrt{x}}) dx = \int \sqrt{1+t} 2tdt$ , como apareció la expresión:  $\sqrt{1+t}$ ; se procede análogamente:  $w = \sqrt{1+t} \Rightarrow t = w^2 - 1, dt = 2wdw$ , esto es:  $\sqrt{1+t} 2tdt = \int w2(w^2-1)2wdw = 4 \int (w^4 - w^2) dw = \frac{4w^5}{5} - \frac{4w^3}{3} + c$

Dado que:  $w = \sqrt{1+t}$ , se tiene:  $= \frac{4(1+t)^{\frac{5}{2}}}{5} - \frac{4(1+t)^{\frac{3}{2}}}{3} + c$

**Respuesta:**  $\int \sqrt{1+\sqrt{x}} dx = \frac{4(1+\sqrt{x})^{\frac{5}{2}}}{5} - \frac{4(1+\sqrt{x})^{\frac{3}{2}}}{3} + c$

**9.6.-Encontrar:**  $\int \frac{dx}{\sqrt{x+1} + \sqrt[4]{x+1}}$

Solución.- Previamente se tiene que el m.c.m. de los índices de las raíces es: 4, por lo cual:  $x+1 = t^4, dx = 4t^3 dt$ , de donde:

$$\begin{aligned} \int \frac{dx}{\sqrt{x+1} + \sqrt[4]{x+1}} &= \int \frac{4t^3 dt}{t^2 + t} = 4 \int \left( t - 1 + \frac{t}{t^2 + t} \right) dt = 4 \int tdt - 4 \int dt + 4 \int \frac{dt}{t+1} \\ &= 2t^2 - 4t + 4\ell\eta|t+1| + c, \text{ dado que: } t = \sqrt[4]{x+1} \end{aligned}$$

Se tiene:  $= 2(x+1)^{\frac{1}{2}} - 4(x+1)^{\frac{1}{2}} + 4\ell\eta|(x+1)^{\frac{1}{2}} + 1| + c$

**Respuesta:**  $\int \frac{dx}{\sqrt{x+1} + \sqrt[4]{x+1}} = 2(x+1)^{\frac{1}{2}} - 4(x+1)^{\frac{1}{2}} + 4\ell\eta|(x+1)^{\frac{1}{2}} + 1| + c$

**9.7.-Encontrar:**  $\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$

Solución.- Previamente se tiene que el m.c.m. de los índices de Las raíces es: 6 , por lo cual:  $x = t^6 \Rightarrow t = \sqrt[6]{x}, dx = 6t^5 dt$ , de donde:

$$\begin{aligned} \int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} &= \int \frac{6t^5 dt}{t^3 + t^2} = 6 \int \frac{t^3 dt}{t+1} = 6 \int \left( t^2 - t + 1 - \frac{1}{t+1} \right) dt = 6 \int t^2 dt - 6 \int t dt + 6 \int dt - 6 \int \frac{dt}{t+1} \\ &= 2t^3 - 3t^2 + 6t - 6 \ln|t+1| + c \end{aligned}$$

Dado que:  $t = \sqrt[6]{x}$

$$\text{Se tiene: } = 2(\sqrt[6]{x})^3 - 3(\sqrt[6]{x})^2 + 6\sqrt[6]{x} - 6 \ln|\sqrt[6]{x} + 1| + c$$

**Respuesta:**  $\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} = 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6 \ln|\sqrt[6]{x} + 1| + c$

**9.8.-Encontrar:**  $\int \frac{dx}{\sqrt{x+1} + \sqrt{(x+1)^3}}$

Solución.- Previamente se tiene igual índice por lo cual:  $\sqrt{x+1} = t \Rightarrow x = t^2 - 1, dx = 2tdt$ , de donde:

$$\int \frac{dx}{\sqrt{x+1} + \sqrt{(x+1)^3}} = \int \frac{2tdt}{t + t^3} = 2 \int \frac{dt}{1+t^2} = 2 \arctan t + c$$

Dado que:  $t = \sqrt{x+1}$  Se tiene:  $= 2 \arctan \sqrt{x+1} + c$

**Respuesta:**  $\int \frac{dx}{\sqrt{x+1} + \sqrt{(x+1)^3}} = 2 \arctan \sqrt{x+1} + c$

**9.9.-Encontrar:**  $\int \frac{\sqrt{x}-1}{\sqrt[3]{x}+1} dx$

Solución.- Previamente se tiene que el m.c.m. de los índices de Las raíces es: 6 , por lo cual:  $x = t^6 \Rightarrow t = \sqrt[6]{x}, dx = 6t^5 dt$ , de donde:

$$\begin{aligned} \int \frac{\sqrt{x}-1}{\sqrt[3]{x}+1} dx &= \int \frac{t^3-1}{t^2+1} 6t^5 dt = 6 \int \frac{t^8-t^5}{t^2+1} dt = 6 \int \left( t^6 - t^4 - t^3 + t^2 + t - 1 - \frac{t-1}{t^2+1} \right) dt \\ &= \frac{6}{7}t^7 - \frac{6}{5}t^5 - \frac{3}{2}t^4 + 2t^3 + 3t^2 - 6t + c_1 - 3 \int \frac{2t-2}{t^2+1} dt \\ &= \frac{6}{7}t^7 - \frac{6}{5}t^5 - \frac{3}{2}t^4 + 2t^3 + 3t^2 - 6t + c_1 - 3 \int \frac{2t-2}{t^2+1} dt + 6 \int \frac{dt}{t^2+1} \\ &= \frac{6}{7}t^7 - \frac{6}{5}t^5 - \frac{3}{2}t^4 + 2t^3 + 3t^2 - 6t - 3 \ln|t^2+1| + 6 \arctan t + c \end{aligned}$$

Dado que:  $t = \sqrt[6]{x}$  , se tiene:

$$= \frac{6}{7}x\sqrt[6]{x} - \frac{6}{5}\sqrt[6]{x^5} - \frac{3}{2}\sqrt[3]{x^2} + 2\sqrt{x} + 3\sqrt[3]{x} - 6\sqrt[6]{x} - 3 \ln|1+\sqrt[3]{x}| + 6 \arctan \sqrt[6]{x} + c$$

**Respuesta:**

$$\int \frac{\sqrt[3]{x-1}}{\sqrt[3]{x+1}} dx = \frac{6}{7} x \sqrt[6]{x} - \frac{6}{5} \sqrt[6]{x^5} - \frac{3}{2} \sqrt[3]{x^2} + 2\sqrt{x} + 3\sqrt[3]{x} - 6\sqrt[6]{x} - 3\ell \eta \left| 1 + \sqrt[3]{x} \right| + 6 \arctan g \sqrt[6]{x} + c$$

**9.10.-Encontrar:**  $\int \frac{\sqrt{x}dx}{x+2}$

Solución.- La expresión “irracional” es  $\sqrt{x}$ , por lo tanto:

$$\sqrt{x} = t \Rightarrow x = t^2, dx = 2tdt,$$

$$\begin{aligned} \text{luego: } \int \frac{\sqrt{x}dx}{x+2} &= \int \frac{t(2tdt)}{t^2+2} = 2 \int \frac{t^2 dt}{t^2+2} = 2 \int \left(1 - \frac{2}{t^2+2}\right) dt = 2 \int dt - 4 \int \frac{dt}{t^2+2} \\ &= 2t - \frac{4}{\sqrt{2}} \arctan g \frac{t}{\sqrt{2}} + c, \text{ dado que: } t = \sqrt{x}, \text{ se tiene: } = 2\sqrt{x} - 2\sqrt{2} \arctan g \sqrt{\frac{x}{2}} + c \end{aligned}$$

**Respuesta:**  $\int \frac{\sqrt{x}dx}{x+2} = 2\sqrt{x} - 2\sqrt{2} \arctan g \sqrt{\frac{x}{2}} + c$

**9.11.-Encontrar:**  $\int \frac{(\sqrt{x+1}+2)dx}{(x+1)^2 - \sqrt{x+1}}$

Solución.- Previamente se tiene igual índice por lo cual:  $\sqrt{x+1} = t \Rightarrow x = t^2 - 1, dx = 2tdt$ , de donde:

$$\begin{aligned} \int \frac{(\sqrt{x+1}+2)dx}{(x+1)^2 - \sqrt{x+1}} &= \int \frac{\left[(x+1)^{\frac{1}{2}} + 2\right]dx}{(x+1)^2 - (x+1)^{\frac{1}{2}}} = \int \frac{t+2}{t^4-t} 2tdt = 2 \int \frac{(t+2)dt}{t(t^3-1)} \\ &= 2 \int \frac{(t+2)dt}{(t-1)(t^2+t+1)} \text{ (*)}, \text{ considerando que:} \end{aligned}$$

$$\frac{t+2}{(t-1)(t^2+t+1)} = \frac{A}{(t-1)} + \frac{Bt+C}{(t^2+t+1)} \Rightarrow A=1, B=-1, C=-1$$

Dado que:  $t = \sqrt{x+1}$  Se tiene:  $= 2 \arctan g \sqrt{x+1} + c$

$$\begin{aligned} \text{(*) } 2 \int \frac{(t+2)dt}{(t-1)(t^2+t+1)} &= 2 \int \frac{dt}{(t-1)} + 2 \int \frac{-t-1}{(t^2+t+1)} dt = 2 \int \frac{dt}{(t-1)} - 2 \int \frac{t+1}{(t^2+t+1)} dt \\ &= 2 \int \frac{dt}{(t-1)} - 2 \int \frac{\frac{1}{2}(2t+1) + \frac{1}{2}}{(t^2+t+1)} dt = 2 \int \frac{dt}{(t-1)} - \int \frac{(2t+1)dt}{(t^2+t+1)} - \int \frac{dt}{(t^2+t+\frac{1}{4})+\frac{3}{4}} \\ &= 2 \int \frac{dt}{(t-1)} - \int \frac{(2t+1)dt}{(t^2+t+\frac{1}{4})+\frac{3}{4}} - \int \frac{dt}{(t^2+t+\frac{1}{4})+\frac{3}{4}} \\ &= 2\ell \eta |t-1| - \ell \eta |t^2+t+1| - \frac{2}{\sqrt{3}} \arctan g \frac{2t+1}{\sqrt{3}} + c \\ &= \ell \eta \left| \frac{(t-1)^2}{(t^2+t+1)} \right| - \frac{2}{\sqrt{3}} \arctan g \frac{2t+1}{\sqrt{3}} + c \end{aligned}$$

Dado que:  $t = \sqrt{x+1}$ , se tiene

**Respuesta:**  $\int \frac{(\sqrt{x+1}+2)dx}{(x+1)^2-\sqrt{x+1}} = \ell\eta \left| \frac{(\sqrt{x+1}-1)^2}{(\sqrt{x+1}+x+2)} \right| - \frac{2}{\sqrt{3}} \arctan g \frac{2\sqrt{x+1}+1}{\sqrt{3}} + c$

## EJERCICIOS PROPUESTOS

**9.12.-**  $\int \frac{1+x}{1+\sqrt{x}} dx$

**9.15.-**  $\int \frac{\sqrt{x+a}}{x+a} dx$

**9.18.-**  $\int \frac{dx}{x-2-\sqrt{x}}$

**9.21.-**  $\int \frac{\sqrt[3]{x+1}}{x} dx$

**9.24.-**  $\int \frac{dx}{\sqrt{x}+\sqrt[4]{x}+2\sqrt[8]{x}}$

**9.13.-**  $\int \frac{1-x}{1+\sqrt{x}} dx$

**9.16.-**  $\int \frac{\sqrt{x}dx}{1+\sqrt[4]{x}}$

**9.19.-**  $\int \frac{\sqrt{1+x}}{\sqrt{1-x}} dx$

**9.22.-**  $\int \frac{\sqrt{a^2-x^2}}{x^3} dx$

**9.25.-**  $\int x^3 \sqrt{x^2+a^2} dx$

**9.14.-**  $\int \frac{dx}{a+b\sqrt{x}}$

**9.17.-**  $\int \frac{\sqrt{x}-\sqrt[6]{x}}{\sqrt[3]{x+1}} dx$

**9.20.-**  $\int \frac{\sqrt{x+a}}{x+b} dx$

**9.23.-**  $\int x^2 \sqrt{x+a} dx$

## RESPUESTAS

**9.12.-**  $\int \frac{1+x}{1+\sqrt{x}} dx$

Solución.- Sea:  $\sqrt{x}=t \Rightarrow x=t^2, dx=2tdt$

$$\begin{aligned} \int \frac{1+x}{1+\sqrt{x}} dx &= \int \frac{1+t^2}{1+t} 2tdt = 2 \int \frac{t+t^3}{1+t} dt = 2 \int \left( t^2 - t + 2 - \frac{2}{t+1} \right) dt \\ &= 2 \int t^2 dt - 2 \int t dt + 4 \int dt - 4 \int \frac{dt}{t+1} = \frac{2t^3}{3} - \frac{2t^2}{2} + 4t - 4\ell\eta |t+1| + c \\ &= \frac{2\sqrt{x^3}}{3} - x + 4\sqrt{x} - 4\ell\eta |\sqrt{x}+1| + c \end{aligned}$$

**9.13.-**  $\int \frac{1-x}{1+\sqrt{x}} dx$

Solución.- Sea:  $\sqrt{x}=t \Rightarrow x=t^2, dx=2tdt$

$$\begin{aligned} \int \frac{1-x}{1+\sqrt{x}} dx &= \int \frac{1-t}{1+t} 2tdt = 2 \int \frac{t-t^2}{1+t} dt = -2 \int t dt + 4 \int dt - 4 \int \frac{dt}{t+1} = -t^2 + 4t - 4\ell\eta |t+1| + c \\ &= -x + 4\sqrt{x} - 4\ell\eta |\sqrt{x}+1| + c \end{aligned}$$

**9.14.-**  $\int \frac{dx}{a+b\sqrt{x}}$

Solución.- Sea:  $\sqrt{x}=t \Rightarrow x=t^2, dx=2tdt$

$$\int \frac{dx}{a+b\sqrt{x}} = \int \frac{2tdt}{a+bt} = 2 \int \frac{tdt}{a+bt} = 2 \int \left( \frac{1}{b} - \frac{a}{b} \frac{1}{a+bt} \right) dt = \frac{2}{b} \int dt - \frac{2a}{b^2} \int \frac{dt}{a+bt}$$

$$= \frac{2}{b} t - \frac{2a}{b^2} \ell \eta |a+bt| + c = \frac{2}{b} \sqrt{x} - \frac{2a}{b^2} \ell \eta |a+b\sqrt{x}| + c$$

**9.15.-**  $\int \frac{\sqrt{x+a}}{x+a} dx$

Solución.- Sea:  $\sqrt{x+a} = t \Rightarrow x = t^2 - a, dx = 2tdt$

$$\int \frac{\sqrt{x+a}}{x+a} dx = \int \frac{t^2 dt}{t^2} = 2 \int dt = 2t + c = 2\sqrt{x+a} + c$$

**9.16.-**  $\int \frac{\sqrt{x}dx}{1+\sqrt[4]{x}}$

Solución.- m.c.m: 4 ; Sea:  $\sqrt[4]{x} = t \Rightarrow x = t^4, dx = 4t^3 dt$

$$\int \frac{\sqrt{x}dx}{1+\sqrt[4]{x}} = \int \frac{t^2 4t^3 dt}{1+t} = 4 \int \frac{t^5 dt}{1+t} = 4 \int \left( t^4 - t^3 + t^2 - t + 1 - \frac{1}{t+1} \right) dt$$

$$= 4 \left( \frac{t^5}{5} - \frac{t^4}{4} + \frac{t^3}{3} - \frac{t^2}{2} + t - \ell \eta |t+1| \right) + c = \frac{4t^5}{5} - t^4 + \frac{4t^3}{3} - 2t^2 + 4t - 4\ell \eta |t+1|$$

$$= \frac{4x^{5/4}}{5} - x + \frac{4x^{3/4}}{3} - 2x^{1/2} + 4x^{1/4} - 4\ell \eta |x^{1/4} + 1|$$

**9.17.-**  $\int \frac{\sqrt{x}-\sqrt[6]{x}}{\sqrt[3]{x}+1} dx$

Solución.- m.c.m: 6 ; Sea:  $\sqrt[6]{x} = t \Rightarrow x = t^6, dx = 6t^5 dt$

$$\int \frac{\sqrt{x}-\sqrt[6]{x}}{\sqrt[3]{x}+1} dx = \int \frac{t^3-t}{t^2+1} 6t^5 dt = 6 \int \frac{(t^8-t^6)dt}{t^2+1} = 6 \int t^6 dt - 2 \int t^4 dt + 2 \int t^2 dt - 2 \int dt + 2 \int \frac{dt}{1+t^2}$$

$$= 6 \left( \frac{t^7}{7} - \frac{2t^5}{5} + \frac{2t^3}{3} - 2t + 2 \arctan t \right) + c = \frac{6t^7}{7} - \frac{12t^5}{5} + 4t^3 - 12t + 12 \arctan t + c$$

$$= \frac{6x^{7/6}}{7} - \frac{12x^{5/6}}{5} + 4x^{3/2} - 12x^{1/6} + 12 \arctan x^{1/6} + c$$

**9.18.-**  $\int \frac{dx}{x-2-\sqrt{x}}$

Solución.- Sea:  $\sqrt{x} = t \Rightarrow x = t^2, dx = 2tdt$

$$\int \frac{dx}{x-2-\sqrt{x}} dx = \int \frac{2tdt}{t^2-2-t} = \int \frac{(2t-1)+1}{t^2-t-2} dt = \int \frac{2t-1}{t^2-t-2} dt + \int \frac{dt}{t^2-t-2}$$

$$= \int \frac{2t-1}{t^2-t-2} dt + \int \frac{dt}{(t-\frac{1}{2})^2 - \frac{9}{4}} = \ell \eta |t^2-t-2| + \frac{1}{\sqrt[3]{2}} \ell \eta \left| \frac{t-\frac{3}{2}}{t+\frac{3}{2}} \right| + c$$

$$= \ell \eta |t^2 - t - 2| + \frac{1}{3} \ell \eta \left| \frac{2t-3}{2t+3} \right| + c = \ell \eta |x - \sqrt{x} - 2| + \frac{1}{3} \ell \eta \left| \frac{2\sqrt{x}-3}{2\sqrt{x}+3} \right| + c$$

**9.19.-**  $\int \sqrt{\frac{1+x}{1-x}} dx$

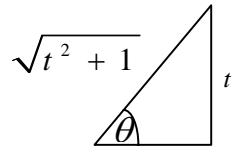
Solución.- Notará el lector, que este caso se diferencia de los anteriores, sin embargo la técnica que se seguirá, tiene la misma fundamentación y la información que se consiga es valiosa. (\*)

Sea:  $\sqrt{\frac{1+x}{1-x}} = t \Rightarrow \frac{1+x}{1-x} = t^2 \Rightarrow 1+x = t^2 - t^2 x \Rightarrow x(1+t^2) = t^2 - 1$

$$x = \frac{t^2 - 1}{t^2 + 1} \Rightarrow dx = \frac{4tdt}{(t^2 + 1)^2}, \text{ luego:}$$

$$(*) \int \sqrt{\frac{1+x}{1-x}} dx = \int \frac{4tdt}{(t^2 + 1)^2} = \int \frac{4t^2 dt}{(t^2 + 1)^2} = 4 \int \frac{t^2 dt}{(\sqrt{t^2 + 1})^4} \quad (**), \quad \text{haciendo uso de}$$

sustituciones trigonométricas convenientes en (\*\*), y de la figura se tiene:



Se tiene:  $t = \tau g \theta, dt = \sec^2 \theta d\theta; \sqrt{t^2 + 1} = \sec \theta$

$$\begin{aligned} (***) 4 \int \frac{t^2 dt}{(\sqrt{t^2 + 1})^4} &= \int \frac{4\tau g^2 \theta \sec^2 \theta d\theta}{\sec^4 \theta} = 4 \int \frac{\tau g^2 \theta}{\sec^2 \theta} d\theta \\ &= 4 \int \sin^2 \theta d\theta = 2 \int d\theta - 2 \int \cos 2\theta d\theta = 2\theta - \sin 2\theta + c = 2\theta - 2 \sin \theta \cos \theta + c \\ &= 2 \arctan gt - 2 \frac{t}{\sqrt{t^2 + 1}} \frac{1}{\sqrt{t^2 + 1}} + c = 2 \arctan gt - \frac{2t}{t^2 + 1} + c = 2 \arctan \tau g \sqrt{\frac{1+x}{1-x}} - \frac{2\sqrt{\frac{1+x}{1-x}}}{\frac{1+x}{1-x} + 1} + c \\ &= 2 \arctan \tau g \sqrt{\frac{1+x}{1-x}} - (1-x) \sqrt{\frac{1+x}{1-x}} + c \end{aligned}$$

**9.20.-**  $\int \frac{\sqrt{x+a}}{x+b} dx$

Solución.- Sea:  $\sqrt{x+a} = t \Rightarrow x = t^2 - a, dx = 2tdt$

$$\begin{aligned} \int \frac{\sqrt{x+a}}{x+b} dx &= \int \frac{t^2 dt}{t^2 - a + b} = 2 \int \frac{t^2 dt}{t^2 + (b-a)} = 2 \int \left( 1 - \frac{b-a}{t^2 + (b-a)} \right) dt \\ &= 2 \int dt - 2(b-a) \int \frac{dt}{t^2 + (b-a)} = 2t - 2(b-a) \frac{1}{\sqrt{b-a}} \arctan \frac{t}{\sqrt{b-a}} + c \end{aligned}$$

$$= 2\sqrt{x+a} - 2\sqrt{b-a} \operatorname{arctg} \sqrt{\frac{x+a}{b-a}} + c$$

$$\mathbf{9.21.-} \int \frac{\sqrt[3]{x+1}}{x} dx$$

Solución.- Sea:  $\sqrt[3]{x+1} = t \Rightarrow x = t^3 - 1, dx = 3t^2 dt$

$$\int \frac{\sqrt[3]{x+1}}{x} dx = \int \frac{t^3 dt}{t^3 - 1} = 3 \int \frac{t^3 dt}{t^3 - 1} = 3 \int \left(1 + \frac{1}{t^3 - 1}\right) dt = 3 \int dt + 3 \int \frac{dt}{t^3 - 1}$$

$$= 3 \int dt + 3 \int \frac{dt}{(t-1)(t^2+t+1)} \text{, por fracciones parciales:}$$

$$\frac{3}{(t-1)(t^2+t+1)} = \frac{A}{(t-1)} + \frac{Bt+C}{(t^2+t+1)} \Rightarrow 3 = A(t^2+t+1) + (Bt+C)(t-1), \text{ de donde:}$$

$A = 1, B = -1, C = -2$ , luego:

$$(*) = 3 \int dt + \int \frac{dt}{t-1} - \int \frac{t+2}{t^2+t+1} dt = 3t + \ell \eta |t-1| - \frac{1}{2} \ell \eta |t^2+t+1| - \sqrt{3} \operatorname{arctg} \left( \frac{2t+1}{\sqrt{3}} \right) + c$$

$$\mathbf{9.22.-} \int \frac{\sqrt{a^2-x^2}}{x^3} dx$$

Solución.- Sea:  $\sqrt{a^2-x^2} = t \Rightarrow x^2 = a^2 - t^2, xdx = -tdt$

$$\int \frac{\sqrt{a^2-x^2}}{x^3} dx = \int \frac{\sqrt{a^2-x^2} x dx}{x^4} = - \int \frac{tdt}{(a^2-t^2)^2} = \int \frac{-t^2 dt}{(a^2-t^2)^2} = \int \frac{-t^2 dt}{(a+t)^2(a-t)^2} \text{ (*)}$$

Por fracciones parciales:

$$\frac{-t^2}{(t+a)^2(t-a)^2} = \frac{A}{(t+a)} + \frac{B}{(t+a)^2} + \frac{C}{(t-a)} + \frac{D}{(t-a)^2}, \text{ de donde:}$$

$A = \frac{1}{4}a, B = -\frac{1}{4}, C = -\frac{1}{4}a, D = -\frac{1}{4}$ , luego:

$$\begin{aligned} (*) \int \frac{-t^2 dt}{(a+t)^2(a-t)^2} &= \frac{1}{4a} \int \frac{dt}{(t+a)} - \frac{1}{4a} \int \frac{dt}{(t+a)^2} - \frac{1}{4a} \int \frac{dt}{(t-a)} - \frac{1}{4a} \int \frac{dt}{(t-a)^2} \\ &= \frac{1}{4a} \ell \eta |(t+a)| + \frac{1}{4(t+a)} - \frac{1}{4a} \ell \eta |(t-a)| + \frac{1}{4(t-a)} + c \\ &= \frac{1}{4a} \ell \eta \left| \frac{(t+a)}{(t-a)} \right| + \frac{1}{4(t+a)} + \frac{1}{4(t-a)} + c \\ &= \frac{1}{4a} \ell \eta \left| \frac{\sqrt{a^2-x^2}+a}{\sqrt{a^2-x^2}-a} \right| + \frac{\sqrt{a^2-x^2}}{2(\cancel{a^2-x^2} \cancel{a^2})} + c = \frac{1}{4a} \ell \eta \left| \frac{\sqrt{a^2-x^2}+a}{\sqrt{a^2-x^2}-a} \right| - \frac{\sqrt{a^2-x^2}}{2x^2} + c \\ &= \frac{1}{4a} \ell \eta \left| \frac{(\sqrt{a^2-x^2}+a)^2}{\cancel{a^2-x^2} \cancel{a^2}} \right| - \frac{\sqrt{a^2-x^2}}{2x^2} + c = \frac{1}{2a} \ell \eta \left| \sqrt{a^2-x^2}+a \right| - \frac{1}{2a} \ell \eta |x| - \frac{\sqrt{a^2-x^2}}{2x^2} + c \end{aligned}$$

$$\mathbf{9.23.-} \int x^2 \sqrt{x+a} dx$$

Solución.- Sea:  $\sqrt{x+a} = t \Rightarrow x = t^2 - a, dx = 2tdt$

$$\begin{aligned}
\int x^2 \sqrt{x+a} dx &= \int (t^2 - a)^2 t 2t dt = 2 \int t^2 (t^2 - a)^2 dt = 2 \int (t^6 - 2at^4 + a^2 t^2) dt \\
&= 2 \int t^6 dt - 4a \int t^4 dt + 2a^2 \int t^2 dt = \frac{2t^7}{7} - \frac{4at^5}{5} + \frac{2a^2 t^3}{3} + c \\
&= \frac{2(x+a)^{\frac{7}{2}}}{7} - \frac{4a(x+a)^{\frac{5}{2}}}{5} + \frac{2a^2(x+a)^{\frac{3}{2}}}{3} + c
\end{aligned}$$

**9.24.** -  $\int \frac{dx}{\sqrt{x} + \sqrt[4]{x} + 2\sqrt[8]{x}}$

Solución.- Sea:  $\sqrt[8]{x} = t \Rightarrow x = t^8, dx = 8t^7 dt$

$$\begin{aligned}
\int \frac{dx}{\sqrt{x} + \sqrt[4]{x} + 2\sqrt[8]{x}} &= \int \frac{8t^7 dt}{t^4 + t^2 + 2t} = 8 \int \frac{t^6 dt}{t^3 + t + 2} = 8 \int \left( t^3 - t - 2 + \frac{t^2 + 4t + 4}{t^3 + t + 2} \right) dt \\
&= 8 \int t^3 dt - 8 \int t dt - 16 \int dt + 8 \int \frac{t^2 + 4t + 4}{t^3 + t + 2} dt = 8 \frac{t^4}{4} - \frac{8t^2}{2} - 16t + 8 \int \frac{t^2 + 4t + 4}{t^3 + t + 2} dt \\
&= 2t^4 - 4t^2 - 16t + 8 \int \frac{t^2 + 4t + 4}{t^3 + t + 2} dt (*) \text{, por fracciones parciales:} \\
\frac{t^2 + 4t + 4}{(t^3 + t + 2)} &= \frac{t^2 + 4t + 4}{(t+1)(t^2 - t + 2)} = \frac{A}{(t+1)} + \frac{Bt + C}{(t^2 - t + 2)} \Rightarrow A = \frac{1}{4}, B = \frac{3}{4}, C = \frac{14}{4}, \text{ luego:} \\
(*) &= 2t^4 - 4t^2 - 16t + 8 \left( \int \frac{\frac{1}{4} dt}{t+1} + \int \frac{\frac{3}{4}t + \frac{14}{4}}{t^2 - t + 2} dt \right) \\
&= 2t^4 - 4t^2 - 16t + 8 \left( \frac{1}{4} \int \frac{dt}{t+1} + \frac{1}{4} \int \frac{3t + 14}{t^2 - t + 2} dt \right) = 2t^4 - 4t^2 - 16t + 2 \int \frac{dt}{t+1} + 2 \int \frac{3t + 14}{t^2 - t + 2} dt \\
&= 2t^4 - 4t^2 - 16t + 2\ell\eta|t+1| + \cancel{2} \int \frac{2t + 28/3 - 31/3 + 31/3}{t^2 - t + 2} dt \\
&= 2t^4 - 4t^2 - 16t + 2\ell\eta|t+1| + 3 \int \frac{(2t-1)}{t^2 - t + 2} dt + 31 \int \frac{dt}{t^2 - t + 2} \\
&= 2t^4 - 4t^2 - 16t + 2\ell\eta|t+1| + 3\ell\eta|t^2 - t + 2| + 31 \int \frac{dt}{(t - 1/2)^2 + 7/4} \\
&= 2t^4 - 4t^2 - 16t + 2\ell\eta|t+1| + 3\ell\eta|t^2 - t + 2| + 31 \frac{2}{\sqrt{7}} \operatorname{arc tg} \frac{t - 1/2}{\sqrt{7}/2} + c \\
&= 2t^4 - 4t^2 - 16t + 2\ell\eta|t+1| + 3\ell\eta|t^2 - t + 2| + \frac{62}{\sqrt{7}} \operatorname{arc tg} \frac{2t-1}{\sqrt{7}} + c \\
&= 2x^{\frac{1}{2}} - 4x^{\frac{3}{4}} - 16x^{\frac{1}{8}} + 2\ell\eta|x^{\frac{1}{4}} + 1| + 3\ell\eta|x^{\frac{1}{4}} - x^{\frac{1}{8}} + 2| + \frac{62}{\sqrt{7}} \operatorname{arc tg} \frac{2x^{\frac{1}{8}} - 1}{\sqrt{7}} + c
\end{aligned}$$

**9.25.** -  $\int x^3 \sqrt{x^2 + a^2} dx$

Solución.- Sea:  $\sqrt{x^2 + a^2} = t \Rightarrow x^2 = t^2 - a^2, x dx = t dt$

$$\begin{aligned}
\int x^3 \sqrt{x^2 + a^2} dx &= \int x^2 \sqrt{x^2 + a^2} x dx = \int (t^2 - a^2) t t dt = \int (t^2 - a^2) t^2 dt = \int (t^4 - a^2 t^2) dt \\
&= \frac{t^5}{5} - \frac{a^2 t^3}{3} + c = \frac{(x^2 + a^2)^{\frac{5}{2}}}{5} - \frac{a^2 (x^2 + a^2)^{\frac{3}{2}}}{3} + c = (x^2 + a^2)^{\frac{5}{2}} \left( \frac{x^2 + a^2}{5} - \frac{a^2}{3} \right) + c \\
&= (x^2 + a^2)^{\frac{5}{2}} \left( \frac{3x^2 - 2a^2}{15} \right) + c
\end{aligned}$$