Cambio de variable

① $$\int \frac{2^{\ln x}}{x}\,dx = \left[\begin{array}{l} \ln x = t \\ \frac{1}{x}\,dx = dt \end{array}\right] = \int 2^t\,dt = \frac{2^t}{\ln 2} + C = \frac{2^{\ln x}}{\ln 2} + C$$

② $$\int 3x\cos x^2\,dx = \left[\begin{array}{l} x^2 = t \\ 2x\,dx = dt \end{array}\right] = \int 3x\cos t\,\frac{dt}{2x} = \frac{3}{2}\int \cos t\,dt = \frac{3}{2}\,\text{sen}\,t =$$

$$dx = \frac{dt}{2x}$$

$$= \frac{3}{2}\,\text{sen}\,x^2 + C$$

③ $$\int \frac{\text{tg}\,x}{\cos^2 x}\,dx = \left[\begin{array}{l} \text{tg}\,x = t \\ \frac{1}{\cos^2 x}\,dx = dt \end{array}\right] = \int t\,dt = \frac{1}{2}t^2 = \frac{\text{tg}^2 x}{2} + C$$

④ $$\int \frac{\text{sen}\,x\cos x}{1+\text{sen}^4 x}\,dx = \left[\begin{array}{l} \text{sen}^2 x = t \\ 2\,\text{sen}\,x\cos x\,dx = dt \end{array}\right] = \frac{1}{2}\int \frac{dt}{1+t^2} = \frac{1}{2}\,\text{arctg}\,t =$$

$$= \frac{1}{2}\,\text{arctg}(\text{sen}^2 x) + C$$

⑤ $$\int \frac{\cos\sqrt{2x}}{\sqrt{2x}}\,dx = \left[\begin{array}{l} \sqrt{2x} = t \\ \frac{1}{\sqrt{2x}}\,dx = dt \end{array}\right] = -\frac{1}{2}\int \cos t\,dt = \text{sen}\,t = \text{sen}\sqrt{2x} + C$$

Por partes

⑥ $$\int x\cos x\,dx = \left[\begin{array}{l} u = x \longrightarrow du = dx \\ dv = \cos x\,dx \longrightarrow v = \text{sen}\,x \end{array}\right] = x\,\text{sen}\,x - \int \text{sen}\,x\,dx = x\,\text{sen}\,x + \cos x + C$$

⑦ $$\int x^2 e^x\,dx = \left[\begin{array}{l} u = x^2 \rightarrow du = 2x\,dx \\ dv = e^x\,dx \rightarrow v = e^x \end{array}\right] = x^2e^x - \int 2xe^x\,dx \overset{①}{=}$$

Calculamos $\int xe^x\,dx$ por partes:

$$\int xe^x\,dx = \left[\begin{array}{l} u = x \rightarrow du = dx \\ dv = e^x\,dx \rightarrow v = e^x \end{array}\right] = xe^x - \int e^x\,dx = xe^x - e^x$$

$$\overset{①}{=} x^2e^x - 2(xe^x - e^x) = e^x(x^2 - 2x + 2) + C$$

⑧ $$\int \cos^2 x\,dx = \int \cos x\cos x\,dx = \left[\begin{array}{l} u = \cos x \longrightarrow du = -\text{sen}\,x\,dx \\ dv = \cos x\,dx \longrightarrow v = \text{sen}\,x \end{array}\right] =$$

$$= \cos x\,\text{sen}\,x - \int -\text{sen}^2\,dx \underset{\text{sen}^2 x + \cos^2 x = 1}{=} \text{sen}\,x\cos x + \int (1-\cos^2 x)\,dx = \text{sen}\,x\cos x + x - \int \cos^2 x\,dx$$

$$\Longrightarrow 2\int \cos^2 x\,dx = \text{sen}\,x\cos x + x \Longrightarrow \int \cos^2 x\,dx = \frac{1}{2}(\text{sen}\,x\cos x + x) + C$$

9) $\int x^2 \cos x\, dx = \left[\begin{array}{l} u = x^2 \rightarrow du = 2x\,dx \\ dv = \cos x\,dx \rightarrow v = \operatorname{sen} x \end{array}\right] = x^2 \operatorname{sen} x - \int 2x \operatorname{sen} x\, dx \underset{(1)}{=}$

Calculamos $\int x \operatorname{sen} x\, dx$ por partes

$$\int x \operatorname{sen} x\, dx = \left[\begin{array}{l} u = x \rightarrow du = dx \\ dv = \operatorname{sen} x\,dx \rightarrow v = -\cos x \end{array}\right] = -x\cos x + \int \cos x\, dx =$$

$$= -x^2 \cos x + \operatorname{sen} x$$

$$\underset{(1)}{=} x^2 \operatorname{sen} x - 2\left(-x^2 \cos x + \operatorname{sen} x\right) = x^2 \operatorname{sen} x + 2x^2 \cos x - 2\operatorname{sen} x + C$$

10) $\int \frac{x}{2} \operatorname{sen} 3x\, dx = \left[\begin{array}{l} \frac{x}{2} = u \rightarrow du = \frac{1}{2}\,dx \\ dv = \operatorname{sen} 3x\,dx \rightarrow -\frac{1}{3}\cos 3x = v \end{array}\right] = -\frac{1}{6} x \cos 3x + \frac{1}{6}\int \cos 3x\, dx$

$$= -\frac{1}{6} x \cos 3x + \frac{1}{18} \operatorname{sen} 3x + C$$

11) $\int x\, 3^x dx = \left[\begin{array}{l} u = x \rightarrow du = dx \\ dv = 3^x dx \rightarrow v = \frac{3^x}{\ln 3} \end{array}\right] = \frac{x\,3^x}{\ln 3} - \int \frac{3^x}{\ln 3} dx =$

$$= \frac{1}{\ln 3}\left(x\,3^x\right) - \frac{1}{\ln 3}\int 3^x dx = \frac{x\,3^x}{\ln 3} - \frac{1}{\ln 3}\frac{3^x}{\ln 3} = \frac{3^x}{\ln 3}\left(x - \frac{1}{\ln 3}\right) + C$$

Integración de funciones racionales

I) Con solo raíces simples en el denominador

12) $\int \frac{4x^2+8x+6}{x^3+2x^2-x-2}\, dx$

$$x^3+2x^2-x-2 = (x-1)(x+1)(x+2)$$

	1	2	−1	−2
1		1	3	2
	1	3	2	0
−1		−1	−2	
	1	2	0	
−2		−2		
	1	0		

$$\frac{4x^2+8x+6}{x^3+2x^2-x-2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2} =$$

$$= \frac{A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1)}{(x-1)(x+1)(x+2)}$$

$$4x^2+8x+6 = A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1) =$$

$$= x^2(A+B+C) + x(3A+B) + (2A-2B-C) \Rightarrow$$

$$\Rightarrow \begin{cases} 4 = A+B+C \\ 8 = 3A+B \\ 6 = 2A-2B-C \end{cases} \rightarrow (A,B,C) = (3,-1,2)$$

También se pueden calcular igualando los numeradores y sustituyendo el valor de las raíces del denominador.

$$\int \frac{4x^2+8x+6}{x^3+2x^2-x-2}\,dx = \int \frac{3}{x-1}\,dx + \int \frac{-1}{x+1}\,dx + \int \frac{2}{x+2}\,dx =$$

$$= 3\ln|x-1| - \ln|x+1| + 2\ln|x+2| + C$$

(13) $\displaystyle\int \frac{x+3}{x^2+3x+2}\,dx$

$x^2+3x+2 = (x+1)(x+2)$

$$\begin{array}{r|rrr} & 1 & 3 & 2 \\ -1 & & -1 & -2 \\ \hline & 1 & 2 & \boxed{0} \end{array}$$

$$\frac{x+3}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$$

$$\left.\begin{array}{l} x=-1 \longrightarrow 2 = A \\ x=-2 \longrightarrow 1 = -B \end{array}\right\} \frac{x+3}{x^2+3x+2} = \frac{2}{x+1} - \frac{1}{x+2}$$

$$\int \frac{x+3}{x^2+3x+2}\,dx = \int \frac{2}{x+1}\,dx - \int \frac{1}{x+2}\,dx = 2\ln|x+1| - \ln|x+2| + C$$

(II) Con raíces múltiples en el denominador

(14) $\displaystyle\int \frac{2x-1}{x^3-3x^2+3x-1}\,dx$

$x^3-3x^2+3x-1 = (x-1)^3$

$$\begin{array}{r|rrrr} & 1 & -3 & 3 & -1 \\ 1 & & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & \boxed{0} \\ 1 & & 1 & -1 & \\ \hline & 1 & -1 & \boxed{0} & \end{array}$$

$$\frac{2x-1}{x^3-3x^2+3x-1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} =$$

$$= \frac{A(x-1)^2+B(x-1)+C}{(x-1)^3}$$

Raíz: $x=1 \Rightarrow C=1$

Valores: $\left\{\begin{array}{l} x=0 \longrightarrow -1 = A-B \\ x=-1 \longrightarrow 3 = A+B \end{array}\right\} \longrightarrow (A,B)=(1,2)$

$$\frac{2x-1}{x^3-3x^2+3x-1} = \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{1}{(x-1)^3}$$

$$\int \frac{2x-1}{x^3-3x^2+3x-1}\,dx = \int \frac{1}{x-1}\,dx + 2\int \frac{1}{(x-1)^2}\,dx + \int \frac{1}{(x-1)^3}\,dx =$$

$$= \ln|x-1| - \frac{2}{x-1} - \frac{1}{2(x-1)^2} + C$$

III Con raíces simples y múltiples en el denominador

15. $\displaystyle\int \frac{3x+7}{x^3-x^2-x+1}\,dx$

$$x^3-x^2-x+1 = (x+1)(x-1)^2$$

	1	−1	−1	1
1		1	0	−1
	1	0	−1	$\boxed{0}$
1		1	1	
	1	1	$\boxed{0}$	

$$\frac{3x+7}{x^3-x^2-x+1} = \frac{A}{x+1} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} =$$

$$= \frac{A(x-1)^2 + B(x+1)(x-1) + C(x+1)}{(x+1)(x-1)^2}$$

Raíces: $x = 1 \longrightarrow C = 5$

$x = -1 \longrightarrow A = 1$

Valor: $x = 0 \longrightarrow B = -1$

$$\frac{3x+7}{x^3-x^2-x+1} = \frac{1}{x+1} - \frac{1}{x-1} + \frac{5}{(x-1)^2}$$

$$\int \frac{3x+7}{x^3-x^2-x+1}\,dx = \int \frac{1}{x+1}\,dx - \int \frac{1}{x-1}\,dx + 5\int \frac{1}{(x-1)^2}\,dx =$$

$$= \ln|x+1| - \ln|x-1| - \frac{5}{x-1} + C$$