

RELACIÓN DE EJERCICIOS DE INTEGRALES.

1. $\int x^5 dx$

$$\int x^5 dx = \frac{x^{5+1}}{5+1} + k = \frac{x^6}{6} + k$$

2. $\int (x + \sqrt{x}) dx$

$$\begin{aligned} \int (x + \sqrt{x}) dx &= \int (x + x^{1/2}) dx = \frac{x^{1+1}}{1+1} + \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + k = \frac{x^2}{2} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + k = \frac{x^2}{2} + \frac{2\sqrt{x^3}}{3} + k = \\ &= \frac{x^2}{2} + \frac{2x\sqrt{x}}{3} + k \end{aligned}$$

3. $\int \left(\frac{3}{\sqrt{x}} - \frac{x\sqrt{x}}{4} \right) dx$ Sol: $6\sqrt{x} - \frac{1}{10}x^2\sqrt{x} + k$

$$\begin{aligned} \int \left(\frac{3}{\sqrt{x}} - \frac{x\sqrt{x}}{4} \right) dx &= \int \left(3x^{-\frac{1}{2}} - \frac{1}{4}x^{\frac{3}{2}} \right) dx = 3 \cdot \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - \frac{1}{4} \cdot \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + k = 3 \cdot \frac{x^{\frac{1}{2}}}{\frac{1}{2}} - \frac{1}{4} \cdot \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + k = \\ &= 6\sqrt{x} - \frac{1}{2} \frac{\sqrt{x^5}}{5} + k = 6\sqrt{x} - \frac{1}{10}x^2\sqrt{x} + k \end{aligned}$$

4. $\int \frac{x^2 dx}{\sqrt{x}}$ Sol: $\frac{2}{5}x^2\sqrt{x} + k$

$$\int \frac{x^2 dx}{\sqrt{x}} = \int \frac{x^2}{x^{\frac{1}{2}}} dx = \int x^{\frac{3}{2}} dx = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + k = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + k = \frac{2\sqrt{x^5}}{5} + k = \frac{2x^2\sqrt{x}}{5} + k$$

5. $\int \left(\frac{1}{x^2} + \frac{4}{x\sqrt{x}} + 2 \right) dx$ Sol: $-\frac{1}{x} - \frac{8}{\sqrt{x}} + 2x + k$

$$\begin{aligned} \int \left(\frac{1}{x^2} + \frac{4}{x\sqrt{x}} + 2 \right) dx &= \int \left(x^{-2} + 4x^{-\frac{3}{2}} + 2 \right) dx = \frac{x^{-2+1}}{-2+1} + 4 \frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + 2x + k = \\ &= \frac{x^{-1}}{-1} + 4 \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + 2x + k = -\frac{1}{x} - 8 \cdot \frac{1}{x^{\frac{1}{2}}} + 2x + k = -\frac{1}{x} - \frac{8}{\sqrt{x}} + 2x + k \end{aligned}$$

6. $\int \frac{dx}{\sqrt[4]{x}}$ Sol: $\frac{4}{3}\sqrt[4]{x^3} + k$

$$\int \frac{dx}{\sqrt[4]{x}} = \int x^{-\frac{1}{4}} dx = \frac{x^{-\frac{1}{4}+1}}{-\frac{1}{4}+1} + k = \frac{x^{\frac{3}{4}}}{\frac{3}{4}} + k = \frac{4x^{\frac{3}{4}}}{3} + k = \frac{4}{3}\sqrt[4]{x^3} + k$$

7. $\int \left(x^2 + \frac{1}{\sqrt[3]{x}}\right)^2 dx$ Sol: $\frac{x^5}{5} + \frac{3}{4}x^2\sqrt[3]{x^2} + 3\sqrt[3]{x} + k$

$$\begin{aligned} \int \left(x^2 + \frac{1}{\sqrt[3]{x}}\right)^2 dx &= \int \left(x^2 + x^{-\frac{1}{3}}\right)^2 dx = \int (x^4 + 2x^{\frac{5}{3}} + x^{-\frac{2}{3}}) dx = \frac{x^5}{5} + 2 \cdot \frac{x^{\frac{8}{3}}}{\frac{8}{3}} + \frac{x^{\frac{1}{3}}}{\frac{1}{3}} + K = \\ &= \frac{1}{5} \cdot x^5 + \frac{3}{4} \cdot x^{\frac{8}{3}} + 3 \cdot x^{\frac{1}{3}} + K = \frac{1}{5} \cdot x^5 + \frac{3}{4} \cdot \sqrt[3]{x^8} + 3 \cdot \sqrt[3]{x} + K = \\ &= \frac{1}{5} \cdot x^5 + \frac{3}{4} \cdot x^2 \cdot \sqrt[3]{x^2} + 3 \cdot \sqrt[3]{x} + K \end{aligned}$$

8. $\int \cos^3 x \cdot \text{sen } x \cdot dx$ Sol: $-\frac{\cos^4 x}{4} + k$

$$\int \cos^3 x \cdot \text{sen } x \cdot dx = - \int \underbrace{\cos^2 x}_{f^3} \cdot \underbrace{(-\text{sen } x)}_{f'} \cdot dx = \frac{\cos^4 x}{4} + k$$

9. $\int x\sqrt{x^2+1} \cdot dx$ Sol: $\frac{1}{3}\sqrt{(x^2+1)^3} + k$

$$\int x\sqrt{x^2+1} \cdot dx = \frac{1}{2} \int \underbrace{2x}_{f'} \cdot \underbrace{(x^2+1)^{\frac{1}{2}}}_{f^{1/2}} dx = \frac{1}{2} \cdot \frac{(x^2+1)^{\frac{3}{2}}}{\frac{3}{2}} + k = \frac{\sqrt{(x^2+1)^3}}{3} + k$$

10. $\int \frac{xdx}{\sqrt{2x^2+3}}$ Sol: $\frac{1}{2}\sqrt{2x^2+3} + k$

$$\int \frac{xdx}{\sqrt{2x^2+3}} = \frac{1}{4} \int \underbrace{4x}_{f'} \cdot \underbrace{(2x^2+3)^{-\frac{1}{2}}}_{f^{-1/2}} dx = \frac{1}{4} \cdot \frac{(2x^2+3)^{\frac{1}{2}}}{\frac{1}{2}} + k = \frac{1}{2}\sqrt{2x^2+3} + k \quad \circ$$

$$\int \frac{xdx}{\sqrt{2x^2+3}} = \frac{2}{4} \int \frac{4xdx}{2\sqrt{2x^2+3}} = \left\{ \int \frac{f'}{2\sqrt{f}} dx = \sqrt{f} + k \right\} = \frac{1}{2}\sqrt{2x^2+3} + k$$

$$11. \int \frac{x^2 dx}{\sqrt{x^3+1}} \quad \text{Sol: } \frac{2}{3}\sqrt{x^3+1} + k$$

$$\int \frac{x^2 dx}{\sqrt{x^3+1}} = \int x^2 (x^3+1)^{-\frac{1}{2}} dx = \frac{1}{3} \int 3x^2 (x^3+1)^{-\frac{1}{2}} dx = \frac{1}{3} \cdot \frac{(x^3+1)^{\frac{1}{2}}}{\frac{1}{2}} + k = \frac{2}{3}\sqrt{x^3+1} + k$$

$$\int \frac{x^2 dx}{\sqrt{x^3+1}} = \frac{2}{3} \int \frac{3x^2 dx}{2\sqrt{x^3+1}} = \left\{ \int \frac{f'}{2\sqrt{f}} dx = \sqrt{f} + k \right\} = \frac{2}{3}\sqrt{x^3+1} + k$$

$$12. \int \frac{\cos x}{\text{sen}^2 x} dx \quad \text{Sol: } -\frac{1}{\text{sen} x} + k$$

$$\int \frac{\cos x}{\text{sen}^2 x} dx = \int \underbrace{\cos x}_{f'} \cdot \underbrace{\text{sen}^{-2} x}_{f^{-2}} dx = \frac{\text{sen}^{-1} x}{-1} + k = -\frac{1}{\text{sen} x} + k$$

$$13. \int x \cdot (x^2+1)^4 dx \quad \text{Sol: } \frac{(x^2+1)^5}{10} + k$$

$$\int x \cdot (x^2+1)^4 dx = \frac{1}{2} \int \underbrace{2x}_{f'} \cdot \underbrace{(x^2+1)^4}_{f^4} dx = \frac{1}{2} \cdot \frac{(x^2+1)^5}{5} + k = \frac{(x^2+1)^5}{10} + k$$

$$14. \int \frac{\text{sen} x}{\cos^3 x} \cdot dx \quad \text{Sol: } \frac{1}{2\cos^2 x} + k$$

$$\int \frac{\text{sen} x}{\cos^3 x} \cdot dx = - \int \underbrace{-\text{sen} x}_{f'} \cdot \underbrace{\cos^{-3} x}_{f^{-3}} dx = -\frac{\cos^{-2} x}{-2} + k = \frac{1}{2\cos^2 x} + k$$

$$15. \int \frac{\text{tg} x}{\cos^2 x} \cdot dx \quad \text{Sol: } \frac{\text{tg}^2 x}{2} + k$$

$$\int \frac{\text{tg} x}{\cos^2 x} \cdot dx = \int \underbrace{\text{tg} x}_{f^1} \cdot \underbrace{\frac{1}{\cos^2 x}}_{f'} \cdot dx = \frac{\text{tg}^2 x}{2} + k$$

$$16. \int \frac{1}{\cos^2 x \sqrt{\text{tg} x - 1}} \cdot dx \quad \text{Sol: } 2\sqrt{\text{tg} x - 1} + k$$

$$\int \frac{1}{\cos^2 x \sqrt{\text{tg} x - 1}} \cdot dx = \int \frac{1}{\cos^2 x} \cdot (\text{tg} x - 1)^{-\frac{1}{2}} \cdot dx = \frac{(\text{tg} x - 1)^{\frac{1}{2}}}{\frac{1}{2}} + k = 2\sqrt{\text{tg} x - 1} + k$$

$$17. \int \frac{L(x+1)}{x+1} dx \quad \text{Sol: } \frac{L^2(x+1)}{2} + k$$

$$\int \frac{L(x+1)}{x+1} dx = \int \underbrace{L(x+1)}_{f'} \cdot \underbrace{\frac{1}{x+1}}_{f'} dx = \frac{L^2(x+1)}{2} + k$$

$$18. \int \frac{\cos x}{\sqrt{2\text{sen } x + 1}} dx \quad \text{Sol: } \sqrt{2\text{sen } x + 1} + k$$

$$\int \frac{\cos x}{\sqrt{2\text{sen } x + 1}} dx = \frac{1}{2} \int \underbrace{2\cos x}_{f'} \underbrace{(2\text{sen } x + 1)^{-\frac{1}{2}}}_{f^{-1/2}} dx = \frac{1}{2} \cdot \frac{(2\text{sen } x + 1)^{\frac{1}{2}}}{\frac{1}{2}} + k = \sqrt{2\text{sen } x + 1} + k$$

$$19. \int \frac{\sqrt{\text{tg } x + 1}}{\cos^2 x} \cdot dx \quad \text{Sol: } \frac{2}{3} \sqrt{(\text{tg } x + 1)^3} + k$$

$$\int \frac{\sqrt{\text{tg } x + 1}}{\cos^2 x} \cdot dx = \int (\text{tg } x + 1)^{\frac{1}{2}} \cdot \frac{1}{\cos^2 x} \cdot dx = \frac{(\text{tg } x + 1)^{\frac{3}{2}}}{\frac{3}{2}} + k = \frac{2}{3} \sqrt{(\text{tg } x + 1)^3} + k$$

$$20. \int \frac{\cos 2x}{(2 + 3\text{sen } 2x)^3} dx \quad \text{Sol: } -\frac{1}{12} \frac{1}{(2 + 3\text{sen } 2x)^2} + k$$

$$\begin{aligned} \int \frac{\cos 2x}{(2 + 3\text{sen } 2x)^3} dx &= \frac{1}{6} \int 6\cos 2x \cdot (2 + 3\text{sen } 2x)^{-3} dx = \frac{1}{6} \cdot \frac{(2 + 3\text{sen } 2x)^{-2}}{-2} + k = \\ &= -\frac{1}{12} \frac{1}{(2 + 3\text{sen } 2x)^2} + k \end{aligned}$$

$$21. \int \frac{\text{sen } 3x}{\sqrt[3]{\cos^4 3x}} dx \quad \text{Sol: } \frac{1}{\sqrt[3]{\cos 3x}} + k$$

$$\int \frac{\text{sen } 3x}{\sqrt[3]{\cos^4 3x}} dx = -\frac{1}{3} \int \underbrace{-3\text{sen } 3x}_{f'} \underbrace{\cos^{-\frac{4}{3}} 3x}_{f^{-4/3}} dx = -\frac{1}{3} \cdot \frac{\cos^{-\frac{1}{3}} 3x}{-\frac{1}{3}} + k = \frac{1}{\sqrt[3]{\cos 3x}} + k$$

$$22. \int \frac{\arcsin x \, dx}{\sqrt{1-x^2}} \quad \text{Sol: } \frac{\arcsin^2 x}{2} + k$$

$$\int \frac{\arcsin x \, dx}{\sqrt{1-x^2}} = \int \arcsin x \cdot \frac{1}{\sqrt{1-x^2}} dx = \frac{\arcsin^2 x}{2} + k$$

$$23. \int \frac{\arcsin^2 x \, dx}{\sqrt{1-x^2}} \quad \text{Sol: } -\frac{\arcsin^3 x}{3} + k$$

$$\int \frac{\arcsin^2 x \, dx}{\sqrt{1-x^2}} = -\int \arcsin^2 x \cdot \frac{-1}{\sqrt{1-x^2}} dx = -\frac{\arcsin^3 x}{3} + k$$

$$24. \int \frac{x}{x^2+1} dx \quad \text{Sol: } \frac{1}{2} \ln(x^2+1) + k$$

$$\int \frac{x}{x^2+1} dx = \frac{1}{2} \int \frac{2x}{x^2+1} dx = \left\{ \int \frac{f'}{f} dx = \ln|f| + k \right\} = \frac{1}{2} \ln(x^2+1) + k$$

$$25. \int \frac{dx}{3x-7} \quad \text{Sol: } \frac{1}{3} \ln|3x-7| + k$$

$$\int \frac{dx}{3x-7} = \frac{1}{3} \int \frac{3}{3x-7} dx = \left\{ \int \frac{f'}{f} dx = \ln|f| + k \right\} = \frac{1}{3} \ln|3x-7| + k$$

$$26. \int \frac{dx}{5-2x} \quad \text{Sol: } -\frac{1}{2} \ln|5-2x| + k$$

$$\int \frac{dx}{5-2x} = -\frac{1}{2} \int \frac{-2}{5-2x} dx = \left\{ \int \frac{f'}{f} dx = \ln|f| + k \right\} = -\frac{1}{2} \ln|5-2x| + k$$

$$27. \int \frac{x+1}{x^2+2x+3} dx \quad \text{Sol: } \frac{1}{2} \ln|x^2+2x+3| + k$$

$$\int \frac{x+1}{x^2+2x+3} dx = \frac{1}{2} \int \frac{2(x+1)}{x^2+2x+3} dx = \frac{1}{2} \int \frac{2x+2}{x^2+2x+3} dx = \frac{1}{2} \ln|x^2+2x+3| + k$$

$$28. \int \tan x \, dx \quad \text{Sol: } -\ln|\cos x| + k$$

$$\int \tan x \cdot dx = \int \frac{\sin x}{\cos x} \cdot dx = -\int \frac{-\sin x}{\cos x} \cdot dx = -\ln|\cos x| + K$$

29. $\int \operatorname{tg} 2x \, dx$ Sol: $-\frac{1}{2} \operatorname{Ln} |\cos 2x| + k$

$$\int \operatorname{tg} 2x \cdot dx = \int \frac{\operatorname{sen} 2x}{\cos 2x} \cdot dx = -\frac{1}{2} \int \frac{-2\operatorname{sen} 2x}{\cos 2x} \cdot dx = -\frac{1}{2} \operatorname{Ln} |\cos 2x| + K$$

30. $\int \operatorname{ctg} x \, dx$ Sol: $\operatorname{Ln} |\operatorname{sen} x| + k$

$$\int \operatorname{ctg} x \cdot dx = \int \frac{\cos x}{\operatorname{sen} x} \cdot dx = \operatorname{Ln} |\operatorname{sen} x| + K$$

31. $\int \frac{dx}{\operatorname{ctg} 3x}$ Sol: $-\frac{1}{3} \operatorname{Ln} |\cos 3x| + k$

$$\int \frac{dx}{\operatorname{ctg} 3x} = \int \operatorname{tg} 3x \cdot dx = \int \frac{\operatorname{sen} 3x}{\cos 3x} \cdot dx = -\frac{1}{3} \int \frac{-3\operatorname{sen} 3x}{\cos 3x} \cdot dx = -\frac{1}{3} \operatorname{Ln} |\cos 3x| + K$$

32. $\int \operatorname{ctg} \frac{x}{3} \, dx$ Sol: $3 \operatorname{Ln} \left| \operatorname{sen} \frac{x}{3} \right| + k$

$$\int \operatorname{ctg} \frac{x}{3} \cdot dx = \int \frac{\cos \frac{x}{3}}{\operatorname{sen} \frac{x}{3}} \cdot dx = 3 \int \frac{\frac{1}{3} \cos \frac{x}{3}}{\operatorname{sen} \frac{x}{3}} \cdot dx = 3 \cdot \operatorname{Ln} \left| \operatorname{sen} \frac{x}{3} \right| + K$$

33. $\int \left(\operatorname{tg} 4x - \operatorname{ctg} \frac{x}{4} \right) dx$ Sol: $-\frac{1}{4} \operatorname{Ln} |\cos 4x| - 4 \operatorname{Ln} \left| \operatorname{sen} \frac{x}{4} \right| + k$

$$\int \left(\operatorname{tg} 4x - \operatorname{ctg} \frac{x}{4} \right) dx = \int \left(\frac{\operatorname{sen} 4x}{\cos 4x} - \frac{\cos \frac{x}{4}}{\operatorname{sen} \frac{x}{4}} \right) dx = \int \frac{\operatorname{sen} 4x}{\cos 4x} dx - \int \frac{\cos \frac{x}{4}}{\operatorname{sen} \frac{x}{4}} dx =$$

$$= -\frac{1}{4} \int \frac{-4\operatorname{sen} 4x}{\cos 4x} dx - 4 \int \frac{\frac{1}{4} \cos \frac{x}{4}}{\operatorname{sen} \frac{x}{4}} dx = -\frac{1}{4} \operatorname{Ln} |\cos 4x| - 4 \operatorname{Ln} \left| \operatorname{sen} \frac{x}{4} \right| + k$$

34. $\int \frac{\cos x}{2\operatorname{sen} x + 3} dx$ Sol: $\frac{1}{2} \operatorname{Ln} (2\operatorname{sen} x + 3) + k$

$$\int \frac{\cos x}{2\operatorname{sen} x + 3} dx = \frac{1}{2} \int \frac{2\cos x}{2\operatorname{sen} x + 3} dx = \frac{1}{2} \operatorname{Ln} (2\operatorname{sen} x + 3) + k$$

$$35. \int \frac{dx}{(1+x^2)\operatorname{arc\,tg} x} \quad \text{Sol: } \mathbf{Ln} \left| \operatorname{arc\,tg} x \right| + k$$

$$\int \frac{dx}{(1+x^2)\operatorname{arc\,tg} x} = \int \frac{1}{\operatorname{arc\,tg} x} \frac{1}{1+x^2} dx = \mathbf{Ln} \left| \operatorname{arc\,tg} x \right| + k$$

$$36. \int \frac{\cos 2x}{2+3\operatorname{sen} 2x} dx \quad \text{Sol: } \frac{1}{6} \mathbf{Ln} \left| 2+3\operatorname{sen} 2x \right| + k$$

$$\int \frac{\cos 2x}{2+3\operatorname{sen} 2x} dx = \frac{1}{6} \int \frac{6\cos 2x}{2+3\operatorname{sen} 2x} dx = \frac{1}{6} \mathbf{Ln} \left| 2+3\operatorname{sen} 2x \right| + k$$

$$37. \int e^{2x} dx \quad \text{Sol: } \frac{1}{2} e^{2x} + k$$

$$\int e^{2x} dx = \frac{1}{2} \int 2e^{2x} dx = \left\{ \int f' e^f dx = e^f + k \right\} = \frac{1}{2} e^{2x} + k$$

$$38. \int e^{\frac{x}{2}} dx \quad \text{Sol: } 2e^{\frac{x}{2}} + k$$

$$\int e^{\frac{x}{2}} dx = 2 \int \frac{1}{2} e^{\frac{x}{2}} dx = \left\{ \int f' e^f dx = e^f + k \right\} = 2e^{\frac{x}{2}} + k$$

$$39. \int e^{\operatorname{sen} x} \cos x dx \quad \text{Sol: } e^{\operatorname{sen} x} + k$$

$$\int e^{\operatorname{sen} x} \cos x dx = \left\{ \int f' e^f dx = e^f + k \right\} = e^{\operatorname{sen} x} + k$$

$$40. \int a^{x^2} \cdot x \cdot dx \quad \text{Sol: } \frac{a^{x^2}}{2\mathbf{L} a} + k$$

$$\int a^{x^2} x \cdot dx = \frac{1}{2\mathbf{L} a} \int \underbrace{a^{x^2} 2x \cdot \mathbf{L} a}_{D(a^{x^2})} dx = \frac{1}{2\mathbf{L} a} a^{x^2} + k$$

$$41. \int e^{\frac{x}{a}} dx \quad \text{Sol: } a e^{\frac{x}{a}} + k$$

$$\int e^{\frac{x}{a}} dx = a \int \frac{1}{a} e^{\frac{x}{a}} dx = a e^{\frac{x}{a}} + k$$

$$42. \int (e^{2x})^2 dx \quad \text{Sol: } \frac{1}{4} e^{4x} + k$$

$$\int (e^{2x})^2 dx = \int e^{4x} dx = \frac{1}{4} \int 4e^{4x} dx = \frac{1}{4} e^{4x} + k$$

$$43. \int 5^x e^x dx \quad \text{Sol: } \frac{5^x e^x}{\ln 5 + 1} + k$$

$$\int 5^x e^x dx = \int (5e)^x dx = \frac{1}{\ln(5e)} \int (5e)^x \ln(5e) dx = \frac{1}{\ln(5e)} (5e)^x + k = \frac{5^x e^x}{\ln 5 + 1} + k$$

$$44. \int (e^{5x} + a^{5x}) dx \quad \text{Sol: } \frac{1}{5} \left(e^{5x} + \frac{a^{5x}}{\ln a} \right) + k$$

$$\begin{aligned} \int (e^{5x} + a^{5x}) dx &= \frac{1}{5} \int 5e^{5x} dx + \frac{1}{5 \ln a} \int 5a^{5x} \ln a dx = \frac{1}{5} e^{5x} + \frac{1}{5 \ln a} \cdot a^{5x} + k = \\ &= \frac{1}{5} \left(e^{5x} + \frac{a^{5x}}{\ln a} \right) + k \end{aligned}$$

$$45. \int e^{x^2+4x+3} (x+2) dx \quad \text{Sol: } \frac{1}{2} e^{x^2+4x+3} + k$$

$$\int e^{x^2+4x+3} (x+2) dx = \frac{1}{2} \int e^{x^2+4x+3} 2(x+2) dx = \frac{1}{2} e^{x^2+4x+3} + k$$

$$46. \int \frac{e^x}{3+4e^x} dx \quad \text{Sol: } \frac{1}{4} \ln(3+4e^x) + k$$

$$\int \frac{e^x}{3+4e^x} dx = \frac{1}{4} \int \frac{4e^x}{3+4e^x} dx = \frac{1}{4} \ln(3+4e^x) + k$$

$$47. \int \cos 5x dx \quad \text{Sol: } \frac{1}{5} \sin 5x + k$$

$$\int \cos 5x dx = \frac{1}{5} \int 5 \cos 5x dx = \left\{ \int f'(x) \cdot \cos f(x) dx = \sin f(x) + k \right\} = \frac{1}{5} \sin 5x + k$$

48. $\int \operatorname{tg}^2 x \, dx$ Sol: $\operatorname{tg} x - x + k$

Por trigonometría sabemos que $\operatorname{tg}^2 x + 1 = \sec^2 x \Rightarrow \operatorname{tg}^2 x = \sec^2 x - 1$, entonces

$$\int \operatorname{tg}^2 x \, dx = \int (\sec^2 x - 1) dx = \int \sec^2 x \, dx - \int dx = \operatorname{tg} x - x + k$$

49. $\int \frac{\cos(\operatorname{Ln}(x))}{x} dx$ Sol: $\operatorname{sen}(\operatorname{Ln}(x)) + k$

$$\int \frac{\cos(\operatorname{Ln}(x))}{x} dx = \cos(\operatorname{Ln}(x)) \frac{1}{x} dx = \operatorname{sen}(\operatorname{Ln}(x)) + k$$

50. $\int \frac{x}{\sqrt{1-x^4}} dx$ Sol: $\frac{1}{2} \operatorname{arc} \operatorname{sen} x^2 + k$

$$\begin{aligned} \int \frac{x}{\sqrt{1-x^4}} dx &= \int \frac{x}{\sqrt{1-(x^2)^2}} dx = \left\{ \int \frac{f'(x)}{\sqrt{1-(f(x))^2}} dx = \left\{ \begin{array}{l} \operatorname{arc} \operatorname{sen} f(x) + k \\ -\operatorname{arccos} f(x) + k \end{array} \right\} = \right. \\ &= \frac{1}{2} \int \frac{2x}{\sqrt{1-(x^2)^2}} dx = \left\{ \int \frac{f'(x) dx}{\sqrt{1-(f(x))^2}} \right\} = \frac{1}{2} \operatorname{arc} \operatorname{sen}(x^2) + k \end{aligned}$$

51. $\int \frac{dx}{\sqrt{1-4x^2}}$ Sol: $\frac{1}{2} \operatorname{arc} \operatorname{sen}(2x) + k$

$$\int \frac{dx}{\sqrt{1-4x^2}} = \int \frac{dx}{\sqrt{1-(2x)^2}} = \frac{1}{2} \int \frac{2dx}{\sqrt{1-(2x)^2}} = \left\{ \int \frac{f'(x) dx}{\sqrt{1-(f(x))^2}} \right\} = \frac{1}{2} \operatorname{arc} \operatorname{sen}(2x) + k$$

52. $\int \frac{dx}{\sqrt{9-4x^2}}$ Sol: $\frac{1}{2} \operatorname{arc} \operatorname{sen}\left(\frac{2x}{3}\right) + k$

$$\begin{aligned} \int \frac{dx}{\sqrt{9-4x^2}} &= \int \frac{dx}{\sqrt{9\left(1-\frac{4x^2}{9}\right)}} = \int \frac{dx}{3\sqrt{1-\left(\frac{2x}{3}\right)^2}} = \frac{1}{3} \int \frac{dx}{\sqrt{1-\left(\frac{2x}{3}\right)^2}} = \frac{1}{3} \cdot \frac{3}{2} \int \frac{\frac{2}{3} dx}{\sqrt{1-\left(\frac{2x}{3}\right)^2}} = \\ &= \frac{1}{2} \int \frac{\frac{2}{3} dx}{\sqrt{1-\left(\frac{2x}{3}\right)^2}} = \left\{ \int \frac{f'(x) dx}{\sqrt{1-(f(x))^2}} \right\} = \frac{1}{2} \operatorname{arc} \operatorname{sen}\left(\frac{2x}{3}\right) + k \end{aligned}$$

$$53. \int \frac{x - \operatorname{arctg} x}{1+x^2} dx \quad \text{Sol: } \frac{1}{2} \operatorname{Ln}(1+x^2) - \frac{1}{2} (\operatorname{arctg} x)^2 + k$$

$$\begin{aligned} \int \frac{x - \operatorname{arctg} x}{1+x^2} dx &= \int \frac{x}{1+x^2} dx - \int \frac{\operatorname{arctg} x}{1+x^2} dx = \frac{1}{2} \int \frac{2x}{1+x^2} dx - \int \operatorname{arctg} x \frac{1}{1+x^2} dx = \\ &= \left\{ \int \frac{f'(x)}{f(x)} dx - \int f^1 \cdot f' dx \right\} = \frac{1}{2} \operatorname{Ln}(1+x^2) - \frac{1}{2} (\operatorname{arctg} x)^2 + k \end{aligned}$$

$$54. \int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} dx \quad \text{Sol: } \frac{4}{3} \sqrt{(1+\sqrt{x})^3} + k$$

$$\begin{aligned} \int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} dx &= \int \frac{1}{\sqrt{x}} \sqrt{1+\sqrt{x}} dx = 2 \int \frac{1}{2\sqrt{x}} (1+\sqrt{x})^{\frac{1}{2}} dx = \left\{ \int f^{\frac{1}{2}} \cdot f' dx \right\} = \\ &= 2 \cdot \frac{(1+\sqrt{x})^{\frac{3}{2}}}{\frac{3}{2}} + k = \frac{4}{3} \sqrt{(1+\sqrt{x})^3} + k \end{aligned}$$

Veamos como podemos realizar esta misma integral por el método de sustitución o cambio de variable.

Haciendo el cambio $1+\sqrt{x}=t$, calculamos dx : $\frac{1}{2\sqrt{x}} dx = dt \Rightarrow dx = 2\sqrt{x} dt$ y

sustituimos en nuestra integral:

$$\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} dx = \int \frac{\sqrt{t}}{\sqrt{x}} \cdot 2\sqrt{x} dt = 2 \int \sqrt{t} dt = 2 \int t^{\frac{1}{2}} dt = 2 \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + k = \frac{4}{3} t^{\frac{3}{2}} + k = \frac{4}{3} \sqrt{t^3} + k =$$

una vez realizada la integral hay que deshacer el cambio de variable y volver a la variable x , con lo que nos quedará:

$$= \frac{4}{3} \sqrt{(1+\sqrt{x})^3} + k$$

$$55. \int \frac{1}{\sqrt{x}\sqrt{1+\sqrt{x}}} dx \quad \text{Sol: } 4\sqrt{1+\sqrt{x}} + k$$

$$\int \frac{1}{\sqrt{x}\sqrt{1+\sqrt{x}}} dx = \int \frac{1}{\sqrt{x}} \frac{1}{\sqrt{1+\sqrt{x}}} dx = 2 \int \frac{1}{2\sqrt{x}} (1+\sqrt{x})^{-\frac{1}{2}} dx = \left\{ \int f^{-\frac{1}{2}} \cdot f' dx \right\} =$$

$$= 2 \cdot \frac{(1 + \sqrt{x})^{\frac{1}{2}}}{\frac{1}{2}} + k = 4\sqrt{1 + \sqrt{x}} + k$$

Veamos como podemos realizar esta misma integral por el método de sustitución o cambio de variable.

Haciendo el cambio $1 + \sqrt{x} = t$, calculamos dx : $\frac{1}{2\sqrt{x}} dx = dt \Rightarrow dx = 2\sqrt{x} dt$ y

sustituimos en nuestra integral:

$$\int \frac{1}{\sqrt{x}\sqrt{1+\sqrt{x}}} dx = \int \frac{1}{\sqrt{x}\sqrt{t}} \cdot 2\sqrt{x} dt = 2 \int \frac{1}{\sqrt{t}} dt = 2 \int t^{-\frac{1}{2}} dt = 2 \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + k = 4t^{\frac{1}{2}} + k = 4\sqrt{t} + k =$$

una vez realizada la integral hay que deshacer el cambio de variable y volver a la variable x , con lo que nos quedará:

$$= 4\sqrt{1 + \sqrt{x}} + k$$

$$56. \int x\sqrt{x-1} dx \quad \text{Sol: } \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + k$$

Hacemos la sustitución $x-1=t^2 \Rightarrow x=t^2+1$

Calculamos la diferencial de x : $dx = 2t dt$ y sustituimos en la integral que deseamos calcular. Tendremos:

$$\begin{aligned} \int x\sqrt{x-1} \cdot dx &= \int (t^2+1)\sqrt{t^2} \cdot 2t dt = 2 \int (t^2+1)t^2 dt = 2 \int (t^4+t^2) dt = 2 \left(\frac{t^5}{5} + \frac{t^3}{3} \right) + k = \\ &= \frac{2}{5} \cdot t^5 + \frac{2}{3} \cdot t^3 + k = \frac{2}{5} \cdot (x-1)^{\frac{5}{2}} + \frac{2}{3} \cdot (x-1)^{\frac{3}{2}} + k \end{aligned}$$

$$57. \int x(5x^2-3)^7 dx \quad \text{Sol: } \frac{1}{80}(5x^2-3)^8 + k$$

Directamente:

$$\begin{aligned} \int x(5x^2-3)^7 dx &= \frac{1}{10} \int 10x(5x^2-3)^7 dx = \left\{ \int f^7 \cdot f' dx \right\} = \frac{1}{10} \cdot \frac{(5x^2-3)^8}{8} + k = \\ &= \frac{1}{80}(5x^2-3)^8 + k \end{aligned}$$

Por sustitución:

Hacemos $5x^2 - 3 = t \Rightarrow 10x dx = dt \Rightarrow dx = \frac{dt}{10x}$ y sustituimos en nuestra integral

$$\int x(5x^2 - 3)^7 dx = \int xt^7 \frac{dt}{10x} = \frac{1}{10} \int t^7 dt = \frac{1}{10} \cdot \frac{t^8}{8} + k = \frac{1}{80} (5x^2 - 3)^8 + k$$

$$58. \int x(2x + 5)^{10} dx \quad \text{Sol: } \frac{1}{4} \left[\frac{(2x + 5)^{12}}{12} - \frac{5(2x + 5)^{11}}{11} \right] + k$$

Por sustitución:

Hacemos $2x + 5 = t \Rightarrow x = \frac{t-5}{2} \Rightarrow dx = \frac{1}{2} dt$ y sustituimos en nuestra integral

$$\begin{aligned} \int x(2x + 5)^{10} dx &= \int \frac{t-5}{2} \cdot t^{10} \cdot \frac{1}{2} dt = \frac{1}{4} \int (t-5)t^{10} dt = \frac{1}{4} \int (t^{11} - 5t^{10}) dt = \\ &= \frac{1}{4} \left[\frac{t^{12}}{12} - 5 \cdot \frac{t^{11}}{11} \right] + k = \frac{1}{4} \left[\frac{(2x + 5)^{12}}{12} - \frac{5(2x + 5)^{11}}{11} \right] + k \end{aligned}$$

$$59. \int xe^x dx \quad \text{Sol: } e^x(x - 1) + k$$

Por el método de integración por partes:

$$\int xe^x dx = \left\{ \begin{array}{l} u = x \Rightarrow du = dx \\ dv = e^x dx \Rightarrow v = e^x \end{array} \right\} = xe^x - \int e^x dx = xe^x - e^x + k = (x - 1)e^x + k$$

$$60. I = \int (x^2 - 3x + 5)e^x dx \quad \text{Sol: } e^x(x^2 - 5x + 10) + k$$

Por el método de integración por partes:

$$\begin{aligned} I &= \int (x^2 - 3x + 5)e^x dx = \left\{ \begin{array}{l} u = x^2 - 3x + 5 \Rightarrow du = (2x - 3)dx \\ dv = e^x dx \Rightarrow v = e^x \end{array} \right\} = \\ &= (x^2 - 3x + 5) e^x - \int e^x(2x - 3)dx = \end{aligned}$$

La integral que nos ha quedado es del mismo tipo que la que pretendemos calcular, por lo que nuevamente aplicaremos el método de integración de partes:

$$\text{Hacemos } \left\{ \begin{array}{l} u = 2x - 3 \Rightarrow du = 2dx \\ dv = e^x dx \Rightarrow v = e^x \end{array} \right\} \text{ y sustituimos:}$$

$$\begin{aligned}
 I &= (x^2 - 3x + 5)e^x - \int e^x(2x - 3)dx = (x^2 - 3x + 5)e^x - \left[(2x - 3)e^x - \int 2e^x dx \right] = \\
 &= (x^2 - 3x + 5)e^x - (2x - 3)e^x + 2 \int e^x dx = (x^2 - 3x + 5)e^x - (2x - 3)e^x + 2e^x + k = \\
 &= [(x^2 - 3x + 5) - (2x - 3) + 2]e^x + k = e^x(x^2 - 5x + 10) + k
 \end{aligned}$$

61. $\int x \operatorname{Ln}(x) dx$ Sol: $\frac{1}{2}x^2 \left(\operatorname{Ln}(x) - \frac{1}{2} \right) + k$

Por el método de integración por partes:

$$\begin{aligned}
 \int x \operatorname{Ln}(x) dx &= \left\{ \begin{array}{l} u = \operatorname{Ln}(x) \Rightarrow du = \frac{1}{x} dx \\ dv = x dx \Rightarrow v = \frac{1}{2} x^2 \end{array} \right\} = \frac{1}{2} x^2 \operatorname{Ln}(x) - \int \frac{1}{2} x^2 \cdot \frac{1}{x} dx = \\
 &= \frac{1}{2} x^2 \operatorname{Ln}(x) - \frac{1}{2} \int x dx = \frac{1}{2} x^2 \operatorname{Ln}(x) - \frac{1}{2} \cdot \frac{1}{2} x^2 + k = \frac{1}{2} x^2 \left(\operatorname{Ln}(x) - \frac{1}{2} \right) + k
 \end{aligned}$$

62. $\int e^{-x} \cos x dx$ Sol: $\frac{1}{2} e^{-x} (\operatorname{sen} x - \cos x) + k$

$$\begin{aligned}
 I &= \int e^{-x} \cos x dx = \left\{ \begin{array}{l} u = e^{-x} \Rightarrow du = -e^{-x} dx \\ dv = \cos x dx \Rightarrow v = \operatorname{sen} x \end{array} \right\} = e^{-x} \operatorname{sen} x - \int -e^{-x} \operatorname{sen} x dx = \\
 &= e^{-x} \operatorname{sen} x + \int e^{-x} \operatorname{sen} x dx
 \end{aligned}$$

Al aplicar el método de partes nos ha quedado una integral del mismo tipo que la que pretendemos calcular, por lo que volvemos a aplicar el mismo método. En ella hacemos:

$$\begin{aligned}
 u &= e^{-x} \Rightarrow du = -e^{-x} dx \\
 dv &= \operatorname{sen} x dx \Rightarrow v = -\cos x
 \end{aligned}$$

Sustituyendo en la expresión anterior nos queda:

$$\begin{aligned}
 I &= e^{-x} \operatorname{sen} x + \int e^{-x} \operatorname{sen} x dx = e^{-x} \operatorname{sen} x + \left[-\cos x \cdot e^{-x} - \int -\cos x \cdot (-e^{-x}) dx \right] = \\
 &= e^{-x} \operatorname{sen} x - \cos x \cdot e^{-x} - \int \cos x \cdot e^{-x} dx
 \end{aligned}$$

es decir, volvemos a la misma integral que pretendemos calcular. Entonces:

$$I = e^{-x} \sin x - \cos x \cdot e^{-x} - I \Rightarrow 2I = e^{-x} \sin x - \cos x \cdot e^{-x} \Rightarrow I = \frac{e^{-x} (\sin x - \cos x)}{2}$$

En consecuencia:

$$I = \int e^{-x} \cos x dx = \frac{e^{-x} (\sin x - \cos x)}{2} + k$$

63. $\int \mathbf{Ln}(1-x) dx$ Sol: $-x - (1-x) \mathbf{Ln}(1-x) + k$

$$\begin{aligned} \int \mathbf{Ln}(1-x) dx &= \left\{ \begin{array}{l} u = \mathbf{Ln}(1-x) \Rightarrow du = \frac{-1}{1-x} dx \\ dv = dx \Rightarrow v = x \end{array} \right\} = x \mathbf{Ln}(1-x) - \int x \cdot \frac{-1}{1-x} dx = \\ &= x \mathbf{Ln}(1-x) - \int \frac{-x}{1-x} dx = x \mathbf{Ln}(1-x) - \int \frac{1-x-1}{1-x} dx = x \mathbf{Ln}(1-x) - \int \left(1 + \frac{-1}{1-x} \right) dx = \\ &= x \mathbf{Ln}(1-x) - (x + \mathbf{Ln}(1-x)) + k = x \mathbf{Ln}(1-x) - x - \mathbf{Ln}(1-x) + k = \\ &= -x - (1-x) \mathbf{Ln}(1-x) + k \end{aligned}$$

64. $\int x^n \mathbf{Ln}(x) dx$ Sol: $\frac{x^{n+1}}{n+1} \left(\mathbf{Ln}(x) - \frac{1}{n+1} \right) + k$

Por el método de integración por partes:

$$\begin{aligned} \int x^n \mathbf{Ln}(x) dx &= \left\{ \begin{array}{l} u = \mathbf{Ln}(x) \Rightarrow du = \frac{1}{x} dx \\ dv = x^n dx \Rightarrow v = \frac{1}{n+1} x^{n+1} \end{array} \right\} = \frac{1}{n+1} x^{n+1} \mathbf{Ln}(x) - \int \frac{1}{n+1} x^{n+1} \cdot \frac{1}{x} dx = \\ &= \frac{1}{n+1} x^{n+1} \mathbf{Ln}(x) - \frac{1}{n+1} \int x^n dx = \frac{1}{n+1} x^{n+1} \mathbf{Ln}(x) - \frac{1}{n+1} \cdot \frac{1}{n+1} x^{n+1} + k = \\ &= \frac{x^{n+1}}{n+1} \left(\mathbf{Ln}(x) - \frac{1}{n+1} \right) + k \end{aligned}$$

65. $\int \arcsen x \, dx$ Sol: $x \arcsen x + \sqrt{1-x^2} + k$

Hacemos el siguiente cambio: $\begin{cases} u = \arcsen x \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = \frac{1}{\sqrt{1-x^2}} \cdot dx \\ v = x \end{cases}$

Sustituyendo en la fórmula de integración por partes obtenemos:

$$\begin{aligned} \int \arcsen x \, dx &= x \cdot \arcsen x - \int x \cdot \frac{1}{\sqrt{1-x^2}} \cdot dx = x \cdot \arcsen x - \int x \cdot (1-x^2)^{-\frac{1}{2}} \cdot dx = \\ &= x \cdot \arcsen x + \frac{1}{2} \int -2x \cdot (1-x^2)^{-\frac{1}{2}} \cdot dx = x \cdot \arcsen x + \frac{1}{2} \cdot \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + k = \\ &= x \cdot \arcsen x + \sqrt{1-x^2} + k \end{aligned}$$

66. $\int \sqrt{1-x^2} \, dx$ Sol: $\frac{1}{2}(\arcsen x + x\sqrt{1-x^2}) + k$

$$\begin{aligned} \int \sqrt{1-x^2} \, dx &= \int \frac{1-x^2}{\sqrt{1-x^2}} \, dx = \int \frac{1}{\sqrt{1-x^2}} \, dx + \int \frac{-x^2}{\sqrt{1-x^2}} \, dx = \\ &= \arcsen x + \int \frac{-x^2}{\sqrt{1-x^2}} \cdot dx = \end{aligned}$$

La integral que nos queda la realizaremos por partes:

$$\begin{aligned} \int \frac{-x^2}{\sqrt{1-x^2}} \cdot dx &= \int x \cdot \frac{-x}{\sqrt{1-x^2}} \cdot dx = \left\{ \begin{array}{l} u = x \Rightarrow du \\ dv = \frac{-x}{\sqrt{1-x^2}} \Rightarrow v = \sqrt{1-x^2} \end{array} \right\} = \\ &= x\sqrt{1-x^2} - \int \sqrt{1-x^2} \, dx \end{aligned}$$

Sustituyendo nos queda:

$$\int \sqrt{1-x^2} \, dx = \arcsen x + \int \frac{-x^2}{\sqrt{1-x^2}} \cdot dx = \arcsen x + x\sqrt{1-x^2} - \int \sqrt{1-x^2} \, dx$$

y se nos repite la misma integral. Entonces:

$$\int \sqrt{1-x^2} \, dx = \arcsen x + x\sqrt{1-x^2} - \int \sqrt{1-x^2} \, dx \Rightarrow$$

$$\Rightarrow 2 \int \sqrt{1-x^2} dx = \text{arc sen } x + x\sqrt{1-x^2} \Rightarrow \int \sqrt{1-x^2} dx = \frac{1}{2}(\text{arc sen } x + x\sqrt{1-x^2}) + k$$

67. $\int x \text{ arc sen } x dx$ Sol: $\frac{1}{4}[(2x^2 - 1)\text{arcsen } x + x\sqrt{1-x^2}] + k$

Hacemos el siguiente cambio: $\begin{cases} u = \text{arc sen } x \\ dv = x dx \end{cases} \Rightarrow \begin{cases} du = \frac{1}{\sqrt{1-x^2}} \cdot dx \\ v = \frac{x^2}{2} \end{cases}$

Sustituyendo en la fórmula de integración por partes obtenemos:

$$\int x \text{ arc sen } x \cdot dx = \frac{x^2}{2} \cdot \text{arc sen } x - \int \frac{x^2}{2} \cdot \frac{1}{\sqrt{1-x^2}} \cdot dx = \frac{x^2}{2} \cdot \text{arc sen } x + \frac{1}{2} \int \frac{-x^2}{\sqrt{1-x^2}} dx =$$

Por el ejercicio anterior tenemos que :

$$\int \frac{-x^2}{\sqrt{1-x^2}} \cdot dx = \int x \cdot \frac{-x}{\sqrt{1-x^2}} \cdot dx = \left\{ \begin{array}{l} u = x \Rightarrow dx \\ dv = \frac{-x}{\sqrt{1-x^2}} dx \Rightarrow v = \sqrt{1-x^2} \end{array} \right\} =$$

$$= x\sqrt{1-x^2} - \int \sqrt{1-x^2} dx = x\sqrt{1-x^2} - \frac{1-x^2}{\sqrt{1-x^2}} dx \Rightarrow$$

$$\int \frac{-x^2}{\sqrt{1-x^2}} dx = x\sqrt{1-x^2} - \int \frac{1}{\sqrt{1-x^2}} dx - \int \frac{-x^2}{\sqrt{1-x^2}} dx = x\sqrt{1-x^2} - \text{arc sen } x - \int \frac{-x^2}{\sqrt{1-x^2}} dx$$

En consecuencia:

$$\int \frac{-x^2}{\sqrt{1-x^2}} dx = x\sqrt{1-x^2} - \text{arc sen } x - \int \frac{-x^2}{\sqrt{1-x^2}} dx$$

Por tanto:

$$2 \int \frac{-x^2}{\sqrt{1-x^2}} dx = x\sqrt{1-x^2} - \text{arc sen } x \Rightarrow \int \frac{-x^2}{\sqrt{1-x^2}} dx = \frac{1}{2}(x\sqrt{1-x^2} - \text{arc sen } x)$$

Sustituyendo obtenemos:

$$\int x \text{ arc sen } x \cdot dx = \frac{x^2}{2} \cdot \text{arc sen } x + \frac{1}{2} \int \frac{-x^2}{\sqrt{1-x^2}} dx =$$

$$\begin{aligned}
&= \frac{x^2}{2} \cdot \arcsin x + \frac{1}{2} \cdot \frac{1}{2} (x\sqrt{1-x^2} - \arcsin x) + k = \\
&= \frac{x^2}{2} \cdot \arcsin x + \frac{1}{4} x\sqrt{1-x^2} - \frac{1}{4} \arcsin x + k = \\
&= \frac{1}{4} [(2x^2 - 1) \arcsin x + x\sqrt{1-x^2}] + k
\end{aligned}$$

68. $\int \arctan x \, dx$ Sol: $x \arctan x - \frac{1}{2} \ln(1+x^2) + k$

$$\begin{aligned}
\int \arctan x \, dx &= \left\{ \begin{array}{l} u = \arctan x \Rightarrow du = \frac{1}{1+x^2} dx \\ dv = dx \Rightarrow v = x \end{array} \right\} = x \cdot \arctan x - \int x \cdot \frac{1}{1+x^2} dx = \\
&= x \cdot \arctan x - \int \frac{x}{1+x^2} dx = x \cdot \arctan x - \frac{1}{2} \int \frac{2x}{1+x^2} dx = x \cdot \arctan x - \frac{1}{2} \ln(1+x^2) + k
\end{aligned}$$

69. $\int \arctan \sqrt{x} \, dx$ Sol: $(x+1) \arctan \sqrt{x} - \sqrt{x} + k$

$$\begin{aligned}
\int \arctan \sqrt{x} \, dx &= \left\{ \begin{array}{l} u = \arctan \sqrt{x} \Rightarrow du = \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}} dx \\ dv = dx \Rightarrow v = x \end{array} \right\} = x \arctan \sqrt{x} - \int x \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}} dx = \\
&= x \arctan \sqrt{x} - \frac{1}{2} \int \frac{\sqrt{x}}{1+x} dx = \left\{ \begin{array}{l} x = t^2 \Rightarrow \\ \Rightarrow dx = 2t dt \end{array} \right\} = x \arctan \sqrt{x} - \frac{1}{2} \int \frac{t}{1+t^2} \cdot 2t dt = \\
&= x \arctan \sqrt{x} - \int \frac{t^2}{1+t^2} dt = x \arctan \sqrt{x} - \int \frac{1+t^2-1}{1+t^2} dt = \\
&= x \arctan \sqrt{x} - \int \frac{1+t^2-1}{1+t^2} dt = x \arctan \sqrt{x} - \int \left(1 - \frac{1}{1+t^2} \right) dt = \\
&= x \arctan \sqrt{x} - \int dt + \int \frac{1}{1+t^2} dt = x \arctan \sqrt{x} - t + \arctan t + k = \\
&= x \arctan \sqrt{x} - \sqrt{x} + \arctan \sqrt{x} + k = (x+1) \arctan \sqrt{x} - \sqrt{x} + k
\end{aligned}$$

$$70. \quad \int \mathbf{Ln} (x + \sqrt{1+x^2}) dx \qquad \text{Sol: } x \mathbf{Ln} (x + \sqrt{1+x^2}) - \sqrt{1+x^2} + k$$

Hacemos: $u = \mathbf{Ln} (x + \sqrt{1+x^2})$ y $dv = dx$ con lo cual

$$\begin{aligned} du &= \frac{1}{x + \sqrt{1+x^2}} \cdot \left(1 + \frac{2x}{2\sqrt{1+x^2}} \right) dx = \frac{1}{x + \sqrt{1+x^2}} \cdot \left(1 + \frac{x}{\sqrt{1+x^2}} \right) dx \Rightarrow \\ \Rightarrow du &= \frac{1}{x + \sqrt{1+x^2}} \cdot \left(\frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}} \right) dx = \frac{1}{\sqrt{1+x^2}} dx \quad y \quad v = x \end{aligned}$$

Sustituyendo en la fórmula de integración por partes, obtenemos:

$$\begin{aligned} \int \mathbf{Ln} (x + \sqrt{1+x^2}) dx &= x \cdot \mathbf{Ln} (x + \sqrt{1+x^2}) - \int x \cdot \frac{1}{\sqrt{1+x^2}} dx = \\ &= x \cdot \mathbf{Ln} (x + \sqrt{1+x^2}) - \int \frac{2x}{2\sqrt{1+x^2}} dx = x \cdot \mathbf{Ln} (x + \sqrt{1+x^2}) - \sqrt{1+x^2} + k \end{aligned}$$

$$71. \quad \int \frac{x \arcsen x}{\sqrt{1-x^2}} dx \qquad \text{Sol: } x - \sqrt{1-x^2} \arcsen x + k$$

$$\begin{aligned} \int \frac{x \arcsen x}{\sqrt{1-x^2}} dx &= \left\{ \begin{array}{l} u = \arcsen x \Rightarrow du = \frac{1}{\sqrt{1-x^2}} dx \\ dv = \frac{x}{\sqrt{1-x^2}} dx \Rightarrow v = -\int \frac{-2x}{2\sqrt{1-x^2}} dx = -\sqrt{1-x^2} \end{array} \right\} = \\ &= -\sqrt{1-x^2} \cdot \arcsen x - \int -\sqrt{1-x^2} \cdot \frac{1}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} \cdot \arcsen x + \int dx = \\ &= -\sqrt{1-x^2} \cdot \arcsen x + x + k \end{aligned}$$

$$72. \int \frac{2x-1}{(x-1)(x-2)} dx \quad \text{Sol: } \text{Ln} \left| \frac{(x-2)^3}{x-1} \right| + k$$

Tenemos una integral de tipo racional donde el grado del numerador es menor que el grado del denominador. Vamos a descomponer el integrando en fracciones simples:

$$(x-1)(x-2) = 0 \Rightarrow \begin{cases} x=1 \\ x=2 \end{cases} \text{ (raíces reales simples)}$$

Entonces:

$$\frac{2x-1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} = \frac{A(x-2) + B(x-1)}{(x-1)(x-2)}$$

Vamos a calcular los coeficientes indeterminados. Al ser los denominadores iguales, los numeradores también lo serán. Por tanto:

$$2x-1 = A(x-2) + B(x-1) \Rightarrow \begin{cases} x=1 \rightarrow 1 = -A \Rightarrow A = -1 \\ x=2 \rightarrow 3 = B \Rightarrow B = 3 \end{cases}$$

Por tanto,

$$\begin{aligned} \int \frac{2x-1}{(x-1)(x-2)} dx &= \int \left(\frac{-1}{x-1} + \frac{3}{x-2} \right) dx = -\int \frac{1}{x-1} dx + 3 \int \frac{1}{x-2} dx = \\ &= -\text{Ln}(x-1) + 3 \text{Ln}(x-2) + k = \text{Ln} \left| \frac{(x-2)^3}{x-1} \right| + k \end{aligned}$$

$$73. \int \frac{xdx}{(x+1)(x+3)(x+5)} \quad \text{Sol: } \frac{1}{8} \text{Ln} \left| \frac{(x+3)^6}{(x+1)(x+5)^5} \right| + k$$

Tenemos una integral de tipo racional donde el grado del numerador es menor que el grado del denominador. Vamos a descomponer el integrando en fracciones simples:

$$(x+1)(x+3)(x+5) = 0 \Rightarrow \begin{cases} x=-1 \\ x=-3 \\ x=-5 \end{cases} \text{ (raíces reales simples)}$$

Entonces:

$$\begin{aligned} \frac{x}{(x+1)(x+3)(x+5)} &= \frac{A}{x+1} + \frac{B}{x+3} + \frac{C}{x+5} = \\ &= \frac{A(x+3)(x+5) + B(x+1)(x+5) + C(x+1)(x+3)}{(x+1)(x+3)(x+5)} \end{aligned}$$

Para calcular los coeficientes indeterminados, al ser los denominadores iguales, los numeradores también lo serán. Por tanto:

$$x = A(x+3)(x+5) + B(x+1)(x+5) + C(x+1)(x+3) \Rightarrow \begin{cases} x = -1 \rightarrow -1 = 8A \Rightarrow A = -\frac{1}{8} \\ x = -3 \rightarrow -3 = -4B \Rightarrow B = \frac{3}{4} \\ x = -5 \rightarrow -5 = 8C \Rightarrow C = -\frac{5}{8} \end{cases}$$

Por tanto,

$$\begin{aligned} \int \frac{x dx}{(x+1)(x+3)(x+5)} &= \int \left(\frac{-\frac{1}{8}}{x+1} + \frac{\frac{3}{4}}{x+3} + \frac{-\frac{5}{8}}{x+5} \right) dx = \\ &= -\frac{1}{8} \int \frac{1}{x+1} dx + \frac{3}{4} \int \frac{1}{x+3} dx - \frac{5}{8} \int \frac{1}{x+5} dx = -\frac{1}{8} \text{Ln}(x+1) + \frac{3}{4} \text{Ln}(x+3) - \frac{5}{8} \text{Ln}(x+5) + k = \\ &= \frac{1}{8} (\text{Ln}(x+1) + 6\text{Ln}(x+3) - 5\text{Ln}(x+5)) + k = \frac{1}{8} \text{Ln} \left| \frac{(x+3)^6}{(x+1)(x+5)^5} \right| + k \end{aligned}$$

$$74. \int \frac{x^5 + x^4 - 8}{x^3 - 4x} dx \quad \text{Sol: } \frac{x^3}{3} + \frac{x^2}{2} + 4x + \text{Ln} \left| \frac{x^2(x-2)^5}{(x+2)^3} \right| +$$

Al ser el grado del numerador mayor que el grado del denominador, antes de aplicar el método de descomposición en fracciones simples tendremos que dividir. De esta forma obtenemos:

$$\frac{x^5 + x^4 - 8}{x^3 - 4x} = x^2 + x + 4 + \frac{4x^2 + 16x - 8}{x^3 - 4x}$$

En consecuencia:

$$\begin{aligned} \int \frac{x^5 + x^4 - 8}{x^3 - 4x} \cdot dx &= \int (x^2 + x + 4) dx + \int \frac{4x^2 + 16x - 8}{x^3 - 4x} \cdot dx = \\ &= \frac{x^3}{3} + \frac{x^2}{2} + 4x + \int \frac{4x^2 + 16x - 8}{x^3 - 4x} \cdot dx \end{aligned}$$

A la integral que nos queda le aplicamos el método de descomposición en fracciones simples. Calculamos las raíces del denominador:

$$x^3 - 4x = 0 \rightarrow x \cdot (x - 4) = 0 \rightarrow \begin{cases} x = 0 \\ x = \pm 2 \end{cases}$$

Entonces:

$$\frac{4x^2 + 16x - 8}{x^3 - 4x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2} = \frac{A(x-2)(x+2) + Bx(x+2) + Cx(x-2)}{x(x-2)(x+2)}$$

Como los denominadores son iguales, los numeradores también lo serán; por tanto:

$$4x^2 + 16x - 8 = A(x-2)(x+2) + Bx(x+2) + Cx(x-2)$$

Calculamos los coeficientes indeterminados: le vamos asignando los valores de las raíces

$$x = 0 \rightarrow -8 = -4A \rightarrow A = 2$$

$$x = 2 \rightarrow 40 = 8B \rightarrow B = 5$$

$$x = -2 \rightarrow -24 = 8C \rightarrow C = -3$$

Por tanto, la fracción descompuesta en fracciones simples nos queda:

$$\frac{4x^2 + 16x - 8}{x^3 - 4x} = \frac{2}{x} + \frac{5}{x-2} - \frac{3}{x+2}$$

La integral de la función pedida será:

$$\begin{aligned} \int \frac{x^3 + x^2 - 8}{x^3 - 4x} \cdot dx &= \frac{x^3}{3} + \frac{x^2}{2} + 4x + \frac{4x^2 + 16x - 8}{x} \cdot dx = \\ &= \frac{x^3}{3} + \frac{x^2}{2} + 4x + \int \left(\frac{2}{x} + \frac{5}{x-2} - \frac{3}{x+2} \right) \cdot dx \\ &= \frac{x^3}{3} + \frac{x^2}{2} + 4x + \int \frac{2}{x} \cdot dx + \int \frac{5}{x-2} \cdot dx - \int \frac{3}{x+2} \cdot dx = \\ &= \frac{x^3}{3} + \frac{x^2}{2} + 4x + 2 \int \frac{1}{x} \cdot dx + 5 \int \frac{1}{x-2} \cdot dx - 3 \int \frac{1}{x+2} \cdot dx = \\ &= \frac{x^3}{3} + \frac{x^2}{2} + 4x + 2 \cdot \text{Ln} |x| + 5 \cdot \text{Ln} |x-2| - 3 \cdot \text{Ln} |x+2| + k = \\ &= \frac{x^3}{3} + \frac{x^2}{2} + 4x + \text{Ln} \left| \frac{x^2(x-2)^5}{(x+2)^3} \right| + \end{aligned}$$

$$75. \int \frac{x^4 dx}{(x^2-1)(x+2)} \quad \text{Sol: } \frac{x^2}{2} - 2x + \frac{1}{6} \ln \left| \frac{(-1)}{(x+1)^3} \right| + \frac{16}{3} \cdot \ln |x+2| + k$$

Como el grado del numerador es mayor que el del denominador, tenemos que dividir, obteniendo:

$$\frac{x^4}{(x^2-1)(x+2)} = x-2 + \frac{5}{(x-1)(x+2)}$$

Con lo que

$$\int \frac{x^4 \cdot dx}{(x^2-1)(x+2)} = \int (x-2) dx + \int \frac{5x^2-4}{(x^2-1)(x+2)} dx = \frac{x^2}{2} - 2x + \int \frac{5x^2-4}{(x^2-1)(x+2)} dx$$

y tendremos que integrar la función racional que nos queda, donde el grado del numerador es menor que el grado del denominador.

Descomponemos en fracciones simples:

$$\frac{5x^2-4}{(x^2-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2} = \frac{A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1)}{(x-1)(x+2)}$$

Como los denominadores son iguales, también lo serán los numeradores. Entonces:

$$5x^2 - 4 = A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1)$$

Calculamos los coeficientes indeterminados:

$$x=1 \quad \rightarrow \quad 1=6A \quad \rightarrow \quad A=\frac{1}{6}$$

$$x=-1 \quad \rightarrow \quad 1=-2B \quad \rightarrow \quad B=-\frac{1}{2}$$

$$x=-2 \quad \rightarrow \quad 16=3C \quad \rightarrow \quad C=\frac{16}{3}$$

Entonces:
$$\frac{5x^2-4}{(x^2-1)(x+2)} = \frac{1}{6} \frac{1}{x-1} + \frac{-1}{2} \frac{1}{x+1} + \frac{16}{3} \frac{1}{x+2}$$

Y, por tanto:

$$\begin{aligned}
\int \frac{x^4 dx}{(x^2-1)(x+2)} &= \frac{x^2}{2} - 2x + \int \frac{5x^2 -}{(x-1)(x+2)} dx = \frac{x^2}{2} - 2x + \int \left(\frac{\frac{1}{6}}{x-1} + \frac{-\frac{1}{2}}{x+1} + \frac{\frac{16}{3}}{x+2} \right) \cdot dx = \\
&= \frac{x^2}{2} - 2x + \int \frac{\frac{1}{6}}{x-1} \cdot dx + \int \frac{-\frac{1}{2}}{x+1} \cdot dx + \int \frac{\frac{16}{3}}{x+2} \cdot dx = \\
&= \frac{x^2}{2} - 2x + \frac{1}{6} \int \frac{1}{x-1} \cdot dx - \frac{1}{2} \int \frac{1}{x+1} \cdot dx + \frac{16}{3} \int \frac{1}{x+2} \cdot dx = \\
&= \frac{x^2}{2} - 2x + \frac{1}{6} \cdot \text{Ln} |x-1| - \frac{1}{2} \cdot \text{Ln} |x+1| + \frac{16}{3} \cdot \text{Ln} |x+2| + k = \\
&= \frac{x^2}{2} - 2x + \frac{1}{6} \cdot (\text{Ln} |x-1| - 3 \cdot \text{Ln} |x+1|) + \frac{16}{3} \cdot \text{Ln} |x+2| + k = \\
&= \frac{x^2}{2} - 2x + \frac{1}{6} \cdot \text{Ln} \left| \frac{-1}{(x+1)^3} \right| + \frac{16}{3} \cdot \text{Ln} |x+2| + k
\end{aligned}$$

76. $\int \frac{dx}{(x-1)^2(x-2)}$ Sol: $\frac{1}{x-1} + \text{Ln} \left| \frac{x-}{ } \right| + k$

Como el grado del numerador es menor que el grado del denominador aplicamos la descomposición en fracciones simples directamente:

$$\begin{aligned}
\frac{1}{(x-1)^2(x-2)} &= \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-2)} = \frac{A(x-1)(x-2) + B(x-2) + C(x-1)^2}{(x-1)^2(x-2)} \rightarrow \\
&\rightarrow 1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2
\end{aligned}$$

Calculamos los coeficientes:

$$x=1: 1 = -B \rightarrow B = -1$$

$$x=2: 1 = C$$

$$x=0: 1 = 2A - 2B + C \rightarrow 1 = 2A + 2 + 1 \rightarrow A = -1$$

Entonces:

$$\int \frac{1}{(x-1)(x-2)} \cdot dx = \int \frac{-1}{x-1} \cdot dx + \int \frac{1}{(x-1)^2} \cdot dx + \int \frac{1}{(x-2)} \cdot dx =$$

$$\begin{aligned}
 &= -\int \frac{1}{x-1} \cdot dx - \int (x-1)^{-2} \cdot dx + \int \frac{1}{x-2} \cdot dx = \\
 &= -\text{Ln} |x-1| - \frac{(x-1)^{-1}}{-1} + \text{Ln} |x-2| + k = \frac{1}{x-1} + \text{Ln} \left| \frac{x-2}{x-1} \right| + k
 \end{aligned}$$

77. $\int \frac{x-8}{x^3-4x^2+4x} dx$ Sol: $\frac{3}{x-2} + \text{Ln} \frac{(x-2)^2}{x^2} + k$

Igual que en el anterior, aplicamos la descomposición en fracciones simples:

Calculamos las raíces del denominador:

$$x^3 - 4x^2 + 4x = 0 \rightarrow x \cdot (x^2 - 4x + 4) = 0 \rightarrow x \cdot (x-2)^2 = 0 \rightarrow \begin{cases} x=0 \\ x=2 \text{ (doble)} \end{cases}$$

Entonces:

$$\begin{aligned}
 \frac{x-8}{x^3-4x^2+4x} &= \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2} = \frac{A(x-2)^2 + Bx(x-2) + Cx}{x(x-2)^2} \Rightarrow \\
 &\Rightarrow x-8 = A(x-2)^2 + Bx(x-2) + Cx \Rightarrow
 \end{aligned}$$

Calculamos los coeficientes:

$$\begin{aligned}
 x=0 &\rightarrow -8 = 4A \rightarrow A = -2 \\
 x=2 &\rightarrow -6 = 2C \rightarrow C = -3 \\
 x=1 &\rightarrow -7 = A - B + C \rightarrow B = 7 + A + C = 7 - 2 - 3 = 2 \rightarrow B = 2
 \end{aligned}$$

Entonces:

$$\begin{aligned}
 \int \frac{x-8}{x^3-4x^2+4x} dx &= \int \left(\frac{-2}{x} + \frac{2}{x-2} + \frac{-3}{(x-2)^2} \right) \cdot dx = \\
 &= -2 \int \frac{1}{x} \cdot dx + 2 \int \frac{1}{x-2} \cdot dx - 3 \int \frac{1}{(x-2)^2} \cdot dx = -2 \text{Ln} |x| + 2 \text{Ln} |x-2| - 3 \int (x-2)^{-2} dx = \\
 &= -2 \text{Ln} |x| + 2 \text{Ln} |x-2| - 3 \cdot \frac{(x-2)^{-1}}{-1} + k = \frac{3}{x-2} + \text{Ln} \frac{(x-2)^2}{x^2} + k
 \end{aligned}$$