

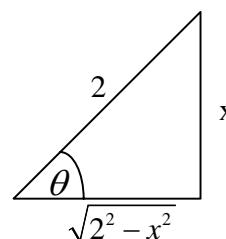
INTEGRACION POR SUSTITUCION TRIGONOMETRICA

Existen integrales que contienen expresiones de las formas: $a^2 - x^2$, $a^2 + x^2$, $x^2 - a^2$, las que tienen fácil solución si se hace la sustitución trigonométrica adecuada. A saber, si la expresión es: $a^2 - x^2$, la sustitución adecuada es: $x = a \operatorname{sen} \theta$ ó $x = a \operatorname{cos} \theta$. Si la expresión es: $a^2 + x^2$, entonces: $x = a \operatorname{sec} \theta$

EJERCICIOS DESARROLLADOS

1. Encontrar: $\int \frac{dx}{\sqrt{(4-x^2)^3}}$

Solución.- Dada la expresión: $4-x^2$, la forma es: $a^2 - x^2$, la sustitución adecuada es: $x = a \operatorname{sen} \theta$ o sea: $x = 2 \operatorname{sen} \theta \therefore dx = 2 \operatorname{cos} \theta d\theta$. Además: $\operatorname{sen} \theta = \frac{x}{a}$. Una figura auxiliar adecuada para ésta situación, es:



$$\begin{aligned} \int \frac{dx}{\sqrt{(4-x^2)^3}} &= \int \frac{dx}{\sqrt{(2^2-x^2)^3}} = \int \frac{2 \operatorname{cos} \theta d\theta}{\sqrt{(2^2-2^2 \operatorname{sen}^2 \theta)^3}} = \int \frac{2 \operatorname{cos} \theta d\theta}{\sqrt{[2^2(1-\operatorname{sen}^2 \theta)]^3}} \\ &= \int \frac{2 \operatorname{cos} \theta d\theta}{\sqrt{(2^2 \operatorname{cos}^2 \theta)^3}} = \int \frac{2 \operatorname{cos} \theta d\theta}{(2 \operatorname{cos} \theta)^3} = \int \frac{2 \operatorname{cos} \theta d\theta}{2^3 \operatorname{cos}^3 \theta} = \frac{1}{2^2} \int \frac{d\theta}{\operatorname{cos}^2 \theta} = \frac{1}{4} \int \sec^2 \theta d\theta \\ &= \frac{1}{4} \int \sec^2 \theta d\theta = \frac{1}{4} \operatorname{tg} \theta + c. \end{aligned}$$

A partir de la figura triangular se tiene:

$$\operatorname{tg} \theta = \frac{x}{\sqrt{4-x^2}}, \text{ Luego: } \frac{1}{4} \operatorname{tg} \theta + c = \frac{1}{4} \frac{x}{\sqrt{4-x^2}} + c$$

Respuesta: $\int \frac{dx}{\sqrt{(4-x^2)^3}} = \frac{1}{4} \frac{x}{\sqrt{4-x^2}} + c$

6.2.-Encontrar: $\int \frac{\sqrt{25-x^2}}{x} dx$

Solución.-

$$\int \frac{\sqrt{25-x^2}}{x} dx = \int \frac{\sqrt{5^2-x^2}}{x} dx, \text{ la forma es: } a^2 - x^2, \text{ luego:}$$

$$\text{Sea: } x = 5 \operatorname{sen} \theta \therefore dx = 5 \cos \theta d\theta, \sqrt{5^2-x^2} = 5 \cos \theta$$

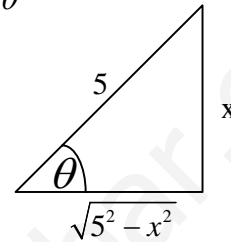
$$\text{Además: } \operatorname{sen} \theta = \frac{x}{5}$$

$$\int \frac{\sqrt{5^2-x^2}}{x} dx = \int \frac{5 \cos \theta d\theta}{5 \operatorname{sen} \theta} = \int \frac{\cos \theta d\theta}{\operatorname{sen} \theta} = \int \frac{(1-\operatorname{sen}^2 \theta) d\theta}{\operatorname{sen} \theta}$$

$$= 5 \int \frac{d\theta}{\operatorname{sen} \theta} - 5 \int \operatorname{sen} \theta d\theta = 5 \int \operatorname{cosec} \theta - 5 \int \operatorname{sen} \theta d\theta$$

$$= 5 \ell \eta |\operatorname{cosec} \theta - \operatorname{cot} \theta| + 5 \cos \theta + c.$$

De la figura se tiene:



$$\operatorname{cosec} \theta = \frac{5}{x}, \operatorname{cot} \theta = \frac{\sqrt{25-x^2}}{x}, \text{ luego:}$$

$$= 5 \ell \eta \left| \frac{5}{x} - \frac{\sqrt{25-x^2}}{x} \right| + \sqrt{25-x^2} + c = 5 \ell \eta \left| \frac{5-\sqrt{25-x^2}}{x} \right| + \sqrt{25-x^2} + c$$

$$\text{Respuesta: } \int \frac{\sqrt{25-x^2}}{x} dx = 5 \ell \eta \left| \frac{5-\sqrt{25-x^2}}{x} \right| + \sqrt{25-x^2} + c$$

$$\text{6.3.-Encontrar: } \int \frac{dx}{\sqrt{(4x-x^2)^3}}$$

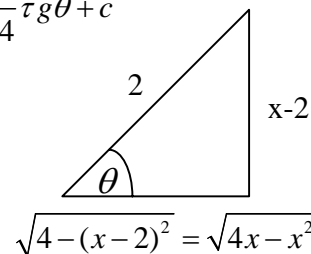
$$\text{Solución.- } 4x-x^2 = -(x^2-4x) = -(x^2-4x+4-4) = 4-(x^2-4x+4) = 2^2-(x-2)^2$$

$$\int \frac{dx}{\sqrt{(4x-x^2)^3}} = \int \frac{dx}{(\sqrt{2^2-(x-2)^2})^3}, \text{ la forma es: } a^2-u^2,$$

$$\text{Luego: } x-2 = 2 \operatorname{sen} \theta \therefore dx = 2 \cos \theta d\theta, \sqrt{2^2-(x-2)^2} = 2 \cos \theta$$

$$\text{Además: } \operatorname{sen} \theta = \frac{x-2}{2}$$

$$\int \frac{dx}{(\sqrt{2^2-(x-2)^2})^3} = \int \frac{2 \cos \theta d\theta}{2^3 \cos^3 \theta} = \frac{1}{4} \int \frac{d\theta}{\cos^2 \theta} = \frac{1}{4} \int \sec^2 \theta d\theta = \frac{1}{4} \operatorname{tg} \theta + c$$



De la figura se tiene:

$$\text{Pero: } \operatorname{tg} \theta = \frac{x-2}{\sqrt{4x-x^2}}, \text{ luego: } \frac{1}{4} \operatorname{tg} \theta + c = \frac{x-2}{4\sqrt{4x-x^2}} + c$$

$$\text{Respuesta: } \int \frac{dx}{\sqrt{(4x-x^2)^3}} = \frac{x-2}{4\sqrt{4x-x^2}} + c$$

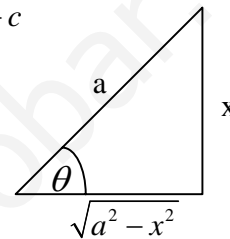
6.4.-Encontrar: $\int \frac{x^2 dx}{(a^2 - x^2)^{3/2}}$

Solución.-

$$\int \frac{x^2 dx}{(a^2 - x^2)^{3/2}} = \int \frac{x^2 dx}{(\sqrt{a^2 - x^2})^3}, \text{ la forma es: } a^2 - x^2$$

Luego: $x = a \operatorname{sen} \theta, dx = a \cos \theta, \sqrt{a^2 - x^2} = a \cos \theta$, además: $\operatorname{sen} \theta = \frac{x}{a}$

$$\begin{aligned} \int \frac{x^2 dx}{(\sqrt{a^2 - x^2})^3} &= \int \frac{a^2 \operatorname{sen}^2 \theta a \cos \theta d\theta}{(a \cos \theta)^3} = \int \frac{\cancel{a^2} \operatorname{sen}^2 \theta \cancel{a} \cos \theta d\theta}{\cancel{a^2} \cancel{a} \cos^2 \theta} = \int \frac{\operatorname{sen}^2 \theta d\theta}{\cos^2 \theta} \\ &= \int \frac{(1 - \cos^2 \theta) d\theta}{\cos^2 \theta} = \int \frac{d\theta}{\cos^2 \theta} - \int d\theta = \int \sec^2 \theta d\theta - \int d\theta = \operatorname{tg} \theta - \theta + c \end{aligned}$$



De la figura se tiene:

Pero: $\operatorname{tg} \theta = \frac{x}{\sqrt{a^2 - x^2}}$, además: $\operatorname{sen} \theta = \frac{x}{a}$ y $\theta = \operatorname{arcsen} \frac{x}{a}$

Luego: $\operatorname{tg} \theta - \theta + c = \frac{x}{\sqrt{a^2 - x^2}} - \operatorname{arcsen} \frac{x}{a} + c$

Respuesta: $\int \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{\sqrt{a^2 - x^2}} - \operatorname{arcsen} \frac{x}{a} + c$

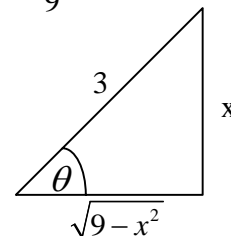
6.5.-Encontrar: $\int \frac{dx}{x^2 \sqrt{9 - x^2}}$

Solución.-

$$\int \frac{dx}{x^2 \sqrt{9 - x^2}} = \int \frac{dx}{x^2 \sqrt{3^2 - x^2}}, \text{ la forma es: } a^2 - x^2$$

Luego: $x = 3 \operatorname{sen} \theta, dx = 3 \cos \theta d\theta, \sqrt{3^2 - x^2} = 3 \cos \theta$, además: $\operatorname{sen} \theta = \frac{x}{3}$

$$\int \frac{dx}{x^2 \sqrt{3^2 - x^2}} = \int \frac{3 \cos \theta d\theta}{3^2 \operatorname{sen}^2 \theta 3 \cos \theta} = \frac{1}{9} \int \frac{d\theta}{\operatorname{sen}^2 \theta} = \frac{1}{9} \int \operatorname{cosec}^2 \theta d\theta = -\frac{1}{9} \operatorname{cot} \theta + c$$



De la figura se tiene:

Pero: $\cot \tau g \theta = \frac{\sqrt{9-x^2}}{x}$, luego: $\frac{1}{9} \cot \tau g \theta + c = -\frac{\sqrt{9-x^2}}{9x} + c$

Respuesta: $\int \frac{dx}{x^2 \sqrt{9-x^2}} = -\frac{\sqrt{9-x^2}}{9x} + c$

6.6.-Encontrar: $\int \frac{x^2 dx}{\sqrt{9-x^2}}$

Solución.-

$\int \frac{x^2 dx}{\sqrt{9-x^2}} = \int \frac{x^2 dx}{\sqrt{3^2-x^2}}$, la forma es: $a^2 - x^2$

Luego: $x = 3 \operatorname{sen} \theta, dx = 3 \cos \theta d\theta, \sqrt{3^2-x^2} = 3 \cos \theta$, además: $\operatorname{sen} \theta = \frac{x}{3}$

Usaremos la misma figura anterior, luego:

$\int \frac{x^2 dx}{\sqrt{3^2-x^2}} = \int \frac{3^2 \operatorname{sen}^2 \theta \cdot 3 \cos \theta d\theta}{3 \cos \theta} = 9 \int \operatorname{sen}^2 \theta d\theta = 9 \int \frac{(1-\cos 2\theta)d\theta}{2}$

$\frac{9}{2} \int \theta - \frac{9}{2} \int \cos 2\theta d\theta = \frac{9}{2} \theta - \frac{9}{4} \operatorname{sen} 2\theta + c = \frac{9}{2} \theta - \frac{9}{4} 2 \operatorname{sen} \theta \cos \theta + c$

$= \frac{9}{2} \theta - \frac{9}{2} \operatorname{sen} \theta \cos \theta + c$, de la figura se tiene que: $\operatorname{sen} \theta = \frac{x}{3}, \cos \theta = \frac{\sqrt{9-x^2}}{3}$ y

$\theta = \operatorname{arcsen} \frac{x}{3}$, luego es equivalente:

$= \frac{9}{2} \operatorname{arcsen} \frac{x}{3} - \frac{9}{4} \frac{x}{3} \frac{\sqrt{9-x^2}}{3} + c = \frac{9}{2} \left(\operatorname{arcsen} \frac{x}{3} - \frac{\sqrt{9-x^2}}{9} \right) + c$

Respuesta: $\int \frac{x^2 dx}{\sqrt{9-x^2}} = \frac{9}{2} \left(\operatorname{arcsen} \frac{x}{3} - \frac{\sqrt{9-x^2}}{9} \right) + c$

6.7.-Encontrar: $\int \sqrt{x^2-4} dx$

Solución.-

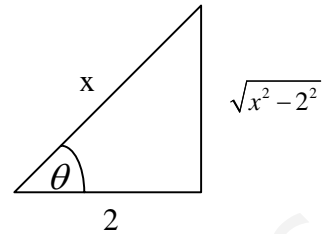
$\int \sqrt{x^2-4} dx = \int \sqrt{x^2-2^2} dx$, la forma es: $x^2 - a^2$

Luego: $x = 2 \sec \theta, dx = 2 \sec \theta \tau g \theta d\theta, \sqrt{x^2-2^2} = 2 \tau g \theta$, además: $\sec \theta = \frac{x}{2}$

$\int \sqrt{x^2-2^2} dx = \int 2 \tau g \theta \cdot 2 \sec \theta \tau g \theta d\theta = 4 \int \sec \theta \tau g^2 \theta d\theta = 4 \int \sec \theta (\sec^2 \theta - 1) d\theta$
 $= 4 \int \sec^3 \theta d\theta - 4 \int \sec \theta d\theta$

Se sabe que: $\int \sec^3 \theta d\theta = \frac{\sec \theta \tau g \theta}{2} + \frac{1}{2} \ln |\sec \theta + \tau g \theta| + c$, luego lo anterior es equivalente a:

$$\begin{aligned}
&= 4 \left(\frac{1}{2} \sec \theta \tau g \theta + \frac{1}{2} \ell \eta |\sec \theta + \tau g \theta| \right) - 4 \ell \eta |\sec \theta + \tau g \theta| + c \\
&= 2 \sec \theta \tau g \theta + 2 \ell \eta |\sec \theta + \tau g \theta| - 4 \ell \eta |\sec \theta + \tau g \theta| + c \\
&= 2 \sec \theta \tau g \theta - 2 \ell \eta |\sec \theta + \tau g \theta| + c
\end{aligned}$$



De la figura se tiene:

$$\sec \theta = \frac{x}{2}, \tau g \theta = \frac{\sqrt{x^2 - 4}}{2}, \text{ luego:}$$

$$\begin{aligned}
&= \cancel{2} \frac{x}{\cancel{2}} \frac{\sqrt{x^2 - 4}}{2} - 2 \ell \eta \left| \frac{x}{2} + \frac{\sqrt{x^2 - 4}}{2} \right| + c = \frac{x \sqrt{x^2 - 4}}{2} - 2 \ell \eta \left| \frac{x + \sqrt{x^2 - 4}}{2} \right| + c \\
&= \frac{x \sqrt{x^2 - 4}}{2} - 2 \ell \eta |x + \sqrt{x^2 - 4}| - 2 \ell \eta 2 + c
\end{aligned}$$

$$\text{Respuesta: } \int \sqrt{x^2 - 4} dx = \frac{x \sqrt{x^2 - 4}}{2} - 2 \ell \eta |x + \sqrt{x^2 - 4}| + c$$

$$\mathbf{6.8.-Encontrar:} \int \frac{x^2 dx}{\sqrt{x^2 - 16}}$$

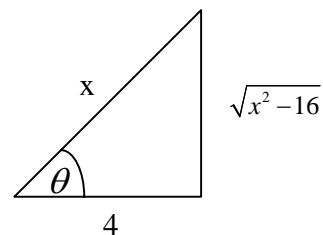
Solución.-

$$\int \frac{x^2 dx}{\sqrt{x^2 - 16}} = \int \frac{x^2 dx}{\sqrt{x^2 - 4^2}}, \text{ la forma es: } x^2 - a^2$$

$$\text{Luego: } x = 4 \sec t, dx = 4 \sec t \tau g t dt, \sqrt{x^2 - 4^2} = 4 \tau g t, \text{ además: } \sec t = \frac{x}{4}$$

$$\int \frac{x^2 dx}{\sqrt{x^2 - 4^2}} = \int \frac{4^2 \sec^2 t (\cancel{4} \sec t \tau g t dt)}{\cancel{4} \tau g t} = 16 \int \sec^3 t dt$$

$$= 16 \left(\frac{1}{2} \sec t \tau g t + \frac{1}{2} \ell \eta |\sec t + \tau g t| + c \right) = 8 \sec t \tau g t + 8 \ell \eta |\sec t + \tau g t| + c$$



De la figura se tiene:

$$\sec t = \frac{x}{4}, \tau g t = \frac{\sqrt{x^2 - 16}}{4}, \text{ luego equivale a:}$$

$$= 8 \frac{x}{4} \frac{\sqrt{x^2 - 16}}{4} + 8 \ell \eta \left| \frac{x}{4} + \frac{\sqrt{x^2 - 16}}{4} \right| + c = \frac{x}{2} \sqrt{x^2 - 16} + 8 \ell \eta \left| \frac{x + \sqrt{x^2 - 16}}{4} \right| + c$$

$$= \frac{x}{2} \sqrt{x^2 - 16} + 8 \ell \eta |x + \sqrt{x^2 - 16}| - 8 \ell \eta 4 + c = \frac{x}{2} \sqrt{x^2 - 16} + 8 \ell \eta |x + \sqrt{x^2 - 16}| + c$$

Respuesta: $\int \frac{x^2 dx}{\sqrt{x^2-16}} = \frac{x}{2} \sqrt{x^2-16} + 8 \ln |x \sqrt{x^2-16}| + c$

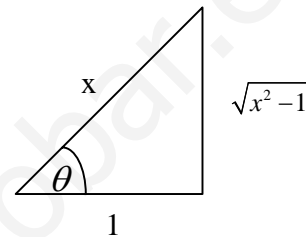
6.9.-Encontrar: $\int \frac{dx}{x\sqrt{x^2-1}}$

Solución.-

$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{dx}{x\sqrt{x^2-1^2}}$, la forma es: $x^2 - a^2$

Luego: $x = \sec t, dx = \sec t \tau g t dt, \sqrt{x^2-1^2} = \tau g t$, además:

$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\sec t \tau g t dt}{\sec t \tau g t} = \int dt = t + c,$



De la figura se tiene:

Dado que: $\sec t = x \Rightarrow t = \arcsin x$, luego:

$t + c = \arcsin x + c$

Respuesta: $\int \frac{dx}{x\sqrt{x^2-1}} = \arcsin x + c$

6.10.-Encontrar: $\int \frac{dx}{(\sqrt{4x^2-24x+27})^3}$

Solución.-

$\int \frac{dx}{(\sqrt{4x^2-24x+27})^3} = \int \frac{dx}{\sqrt{4(x^2-6x+27/4)}^3} = \int \frac{dx}{\sqrt{4^3} (\sqrt{x^2-6x+27/4})^3}$

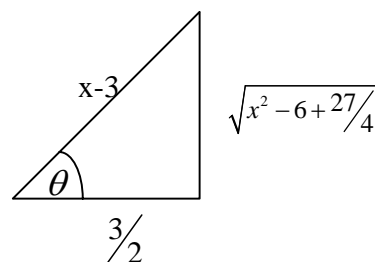
$= \frac{1}{8} \int \frac{dx}{\sqrt{(x^2-6x+27/4)}^3}$, Se tiene:

$x^2 - 6x + \frac{27}{4} = (x^2 - 6x + \underline{\quad}) + \frac{27}{4} - \underline{\quad} = (x^2 - 6x + 9) + \frac{27}{4} - 9$

$= (x^2 - 6x + 9) - \frac{9}{4} = (x^2 - 6x + \frac{27}{4}) = (x-3)^2 - (\frac{3}{2})^2$, la expresión anterior equivale a:

$\frac{1}{8} \int \frac{dx}{(\sqrt{x^2-6x+27/4})^3} = \frac{1}{8} \int \frac{dx}{[\sqrt{(x-3)^2 - (\frac{3}{2})^2}]^3}$, siendo la forma: $u^2 - a^2$, luego:

$x-3 = \frac{3}{2} \sec t, dx = \frac{3}{2} \sec t \tau g t dt$, además: $\sec t = \frac{x-3}{\frac{3}{2}}$



De la figura se tiene:

$$\sec t = \frac{x}{4}, \tau g t = \frac{\sqrt{x^2 - 16}}{4}, \text{ luego equivale a:}$$

$$\begin{aligned} \frac{1}{8} \int \frac{dx}{\left[\sqrt{(x-3)^2 - (3/2)^2} \right]^3} &= \frac{1}{8} \int \frac{3/2 \sec t \tau g t dt}{(3/2)^2 \tau g^3 t} = \frac{1}{8} \frac{1}{3^2} \int \frac{\sec t dt}{\tau g^2 t} = \frac{1}{18} \int \frac{1}{\frac{\cos t}{\sin^2 t} \cos^2 t} \\ &= \frac{1}{18} \int \frac{\cos t dt}{(\sin t)^2} = \frac{1}{18} \int (\sin t)^{-2} \cos t dt = \frac{1}{18} \frac{(\sin t)^{-1}}{-1} + c = -\frac{1}{18} \frac{1}{(\sin t)} + c \\ &= -\frac{1}{18} \cos e c t + c, \text{ como: } \cos e c t = \frac{x-3}{\sqrt{x^2 - 6x + 27/4}}, \text{ entonces:} \\ &= -\frac{1}{18} \frac{x-3}{\sqrt{x^2 - 6x + 27/4}} + c = -\frac{1}{18} \frac{x-3}{\sqrt{\frac{4x^2 - 24x + 27}{4}}} + c = -\frac{1}{18} \frac{x-3}{\frac{\sqrt{4x^2 - 24x + 27}}{2}} + c \\ &= -\frac{1}{9} \frac{x-3}{\sqrt{4x^2 - 24x + 27}} + c \end{aligned}$$

$$\text{Respuesta: } \int \frac{dx}{(\sqrt{4x^2 - 24x + 27})^3} = -\frac{1}{9} \frac{x-3}{\sqrt{4x^2 - 24x + 27}} + c$$

$$\mathbf{6.11.-Encontrar:} \int \frac{dx}{\sqrt{(16+x^2)^4}}$$

Solución.-

$$\int \frac{dx}{\sqrt{(16+x^2)^4}} = \int \frac{dx}{\sqrt{(4^2+x^2)^4}}$$

$$\text{Luego: } x = 4\tau g t, dx = 4\sec^2 t dt, \sqrt{4^2+x^2} = 4\sec t, \text{ además: } \tau g t = \frac{x}{4}$$

$$\begin{aligned} \int \frac{dx}{\sqrt{(4^2+x^2)^4}} &= \int \frac{4\sec^2 t dt}{4^4 \sec^4 t} = \frac{1}{64} \int \frac{dt}{\sec^2 t} = \frac{1}{64} \int \cos^2 t dt = \frac{1}{64} \int \frac{(1+\cos 2t)}{2} dt \\ &= \frac{1}{128} \int dt + \frac{1}{128} \int \cos 2t dt = \frac{1}{128} t + \frac{1}{256} \sin 2t + c \end{aligned}$$

Como: $\tau g t = \frac{x}{4} \Rightarrow t = \text{arc } \tau g \frac{x}{4}$, $\sin 2t = 2\sin t \cos t$; luego:

$$\frac{1}{128} t + \frac{1}{256} \sin 2t + c = 2 \frac{x}{\sqrt{16+x^2}} \frac{4}{\sqrt{16+x^2}} = \frac{8x}{16+x^2}, \text{ Se tiene:}$$

$$\frac{1}{128} \text{arc } \tau g \frac{x}{4} + \frac{1}{256} \frac{8x}{16+x^2} + c = \frac{1}{128} \text{arc } \tau g \frac{x}{4} + \frac{x}{32(16+x^2)} + c$$

Respuesta: $\int \frac{dx}{\sqrt{(16+x^2)^4}} = \frac{1}{128} \operatorname{arctg} \frac{x}{4} + \frac{x}{32(16+x^2)} + c$

6.12.-Encontrar: $\int \frac{x^2 dx}{(x^2+100)^{3/2}}$

Solución.-

$$\int \frac{x^2 dx}{(x^2+100)^{3/2}} = \int \frac{x^2 dx}{(\sqrt{x^2+10^2})^3},$$

se tiene: $x = 10 \operatorname{tg} t, dt = 10 \sec^2 t dt, \sqrt{x^2+10^2} = 10 \sec t$; además: $\operatorname{tg} t = \frac{x}{10}$, luego:

$$\begin{aligned} \int \frac{x^2 dx}{(\sqrt{x^2+10^2})^3} &= \int \frac{10^2 \operatorname{tg}^2 t (10 \sec^2 t) dt}{(10^3 \sec^3 t)} = \int \frac{\operatorname{tg}^2 t dt}{\sec t} = \int \frac{\operatorname{sen}^2 t}{\cos t} dt = \int \frac{\operatorname{sen}^2 t}{\cos t} dt \\ &= \int \frac{(1-\cos^2 t)}{\cos t} dt = \int \frac{dt}{\cos t} - \int \cos t dt = \int \sec t dt - \int \cos t dt = \ell \eta |\sec t + \operatorname{tg} t| - \operatorname{sen} t + c \end{aligned}$$

Como: $\sec t = \frac{\sqrt{100+x^2}}{10}, \operatorname{tg} t = \frac{x}{10}$, además: $\operatorname{sen} t = \frac{x}{\sqrt{100+x^2}}$

$$\begin{aligned} &= \ell \eta \left| \frac{\sqrt{100+x^2}}{10} + \frac{x}{10} \right| - \frac{x}{\sqrt{x^2+100}} + c = \ell \eta \left| \frac{\sqrt{100+x^2} + x}{10} \right| - \frac{x}{\sqrt{x^2+100}} + c \\ &= \ell \eta \left| \sqrt{100+x^2} + x \right| - \frac{x}{\sqrt{x^2+100}} - \ell \eta 10 + c = \ell \eta \left| \sqrt{100+x^2} + x \right| - \frac{x}{\sqrt{x^2+100}} + c \end{aligned}$$

Respuesta: $\int \frac{x^2 dx}{(x^2+100)^{3/2}} = \ell \eta \left| \sqrt{100+x^2} + x \right| - \frac{x}{\sqrt{x^2+100}} + c$

Nota: En los ejercicios 6.11 y 6.12 se ha omitido la figura (triángulo rectángulo). Conviene hacerla y ubicar los datos pertinentes. En adelante se entenderá que el estudiante agregará este complemento tan importante.

6.13.-Encontrar: $\int \frac{x^2 dx}{(x^2+8^2)^{3/2}}$

Solución.-

$$\int \frac{x^2 dx}{(x^2+8^2)^{3/2}} = \int \frac{x^2 dx}{(\sqrt{x^2+8^2})^3},$$

se tiene: $x = 8 \operatorname{tg} t, dt = 8 \sec^2 t dt, \sqrt{x^2+8^2} = 8 \sec t$ además: $\operatorname{tg} t = \frac{x}{8}$, luego:

$$\int \frac{x^2 dx}{(\sqrt{x^2+8^2})^3} = \int \frac{8^2 \operatorname{tg}^2 t (8 \sec^2 t) dt}{8^3 \sec^3 t} = \int \frac{\operatorname{tg}^2 t dt}{\sec t} = \int \sec t dt - \int \cos t dt$$

$$= \ell \eta |\sec t + \tau g t| - \operatorname{sen} t + c, \text{ como: } \sec t = \frac{\sqrt{x^2 + 64}}{8}, \tau g t = \frac{x}{8}, \operatorname{sen} t = \frac{x}{\sqrt{x^2 + 64}}$$

Se tiene como expresión equivalente:

$$= \ell \eta \left| \frac{\sqrt{x^2 + 64}}{8} + \frac{x}{8} \right| - \frac{x}{\sqrt{x^2 + 64}} + c = \ell \eta \left| \frac{\sqrt{x^2 + 64} + x}{8} \right| - \frac{x}{\sqrt{x^2 + 64}} + c$$

$$= \ell \eta \left| \sqrt{x^2 + 64} + x \right| - \frac{x}{\sqrt{x^2 + 64}} + c$$

Respuesta: $\int \frac{x^2 dx}{(x^2 + 8^2)^{3/2}} = \ell \eta \left| \sqrt{x^2 + 64} + x \right| - \frac{x}{\sqrt{x^2 + 64}} + c$

6.14.-Encontrar: $\int \frac{dx}{(\sqrt{3^2 + x^2})^4}$

Solución.- se tiene: $x = 3 \tau g t, dx = 3 \sec^2 t dt, \sqrt{3^2 + x^2} = 3 \sec t$, además:

$$\tau g t = \frac{x}{3}$$

$$\int \frac{dx}{(\sqrt{3^2 + x^2})^4} = \int \frac{\cancel{3} \sec^2 t dt}{3^4 + \sec^4 t} = \frac{1}{3^3} \int \frac{dt}{\sec^2 t} = \frac{1}{27} \int \cos^2 t dt = \frac{1}{54} t + \frac{1}{54} \int \cos 2t dt$$

$$= \frac{1}{54} t + \frac{1}{108} \operatorname{sen} 2t + c_1 = \frac{1}{54} t + \frac{1}{108} 2 \operatorname{sen} t \cos t + c = \frac{1}{54} t + \frac{1}{54} \operatorname{sen} t \cos t + c$$

Como: $\tau g t = \frac{x}{3} \Rightarrow t = \operatorname{arc} \tau g \frac{x}{3}$, además: $\operatorname{sen} t = \frac{x}{\sqrt{9 + x^2}}, \cos t = \frac{3}{\sqrt{9 + x^2}}$

$$= \frac{1}{54} \operatorname{arc} \tau g \frac{x}{3} + \frac{1}{54} \frac{x}{\sqrt{9 + x^2}} \frac{3}{\sqrt{9 + x^2}} + c = \frac{1}{54} \operatorname{arc} \tau g \frac{x}{3} + \frac{x}{18(9 + x^2)} + c$$

Respuesta: $\int \frac{dx}{(\sqrt{3^2 + x^2})^4} = \frac{1}{54} \operatorname{arc} \tau g \frac{x}{3} + \frac{x}{18(9 + x^2)} + c$

6.15.-Encontrar: $\int \frac{dx}{\sqrt{x^2 - 4x + 13}}$

Solución.- Completando cuadrados se tiene:

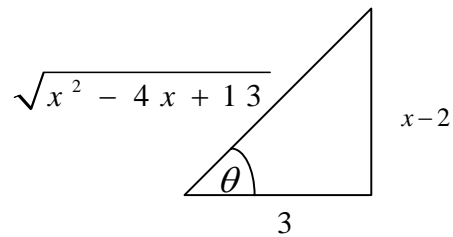
$$x^2 - 4x + 13 = (x^2 - 4x + \underline{\quad}) + 13 - \underline{\quad} = (x^2 - 4x + 4) + 13 - 4 = (x - 2)^2 + 3^2$$

Se tiene: $x - 2 = 3 \tau g t, dx = 3 \sec^2 t dt, \sqrt{3^2 + x^2} = 3 \sec t$

$$\sqrt{(x - 2)^2 + 3^2} = \sqrt{x^2 - 4x + 13} = 3 \sec t,$$

Sea: $x - 2 = 3 \tau g t, dx = 3 \sec^2 t dt$; además: $\tau g t = \frac{x - 2}{3}$, luego:

$$\int \frac{dx}{\sqrt{(x - 2)^2 + 3^2}} = \int \frac{\cancel{3} \sec^2 t dt}{3 \sec t} = \int \sec t dt = \ell \eta |\sec t + \tau g t| + c$$



De la figura se tiene:

$$\sec t = \frac{\sqrt{x^2 - 4x + 13}}{3}, \tau gt = \frac{x-2}{3}, \text{ luego:}$$

$$= \ell \eta \left| \frac{\sqrt{x^2 - 4x + 13}}{3} + \frac{x-2}{3} \right| + c = \ell \eta \left| \frac{\sqrt{x^2 - 4x + 13} + (x-2)}{3} \right| + c$$

$$= \ell \eta \left| \sqrt{x^2 - 4x + 13} + (x-2) \right| + c$$

Respuesta: $\int \frac{dx}{\sqrt{x^2 - 4x + 13}} = \ell \eta \left| \sqrt{x^2 - 4x + 13} + (x-2) \right| + c$

6.16.-Encontrar: $\int \sqrt{1+4x^2} dx$

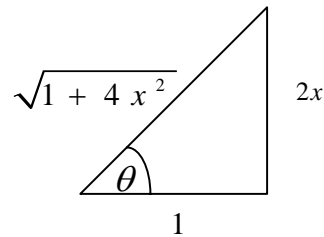
Solución.-

$$\int \sqrt{1+4x^2} dx = \int \sqrt{1^2 + (2x)^2} dx$$

Se tiene: $2x = \tau gt, 2dx = \sec^2 t dt \Rightarrow dx = \frac{1}{2} \sec^2 t dt$, Además: $\tau gt = \frac{2x}{1}$

$$\int \sqrt{1^2 + (2x)^2} dx = \int \sqrt{1^2 + \tau g^2 t} \frac{1}{2} \sec^2 t dt = \frac{1}{2} \int \sec t \sec^2 t dt = \frac{1}{2} \int \sec^3 t dt$$

$$= \frac{1}{4} \sec t \tau gt + \frac{1}{4} \ell \eta |\sec t \tau gt| + c,$$



De la figura se tiene:

$$\sec t = \frac{\sqrt{1+4x^2}}{1}, \tau gt = 2x$$

$$= \frac{1}{4} \sqrt{1+4x^2} 2x + \frac{1}{4} \ell \eta \left| \sqrt{1+4x^2} \right| + 2x + c$$

Respuesta: $\int \sqrt{1+4x^2} dx = \frac{1}{4} \sqrt{1+4x^2} 2x + \frac{1}{4} \ell \eta \left| \sqrt{1+4x^2} \right| + 2x + c$

EJERCICIOS PROPUESTOS:

Utilizando esencialmente la técnica de sustitución por variables trigonométricas, encontrar las integrales siguientes:

6.17.- $\int \sqrt{4-x^2}$

6.18.- $\int \frac{dx}{\sqrt{a^2-x^2}}$

6.19.- $\int \frac{dx}{x^2+a^2}$

$$6.20.- \int \frac{dx}{x^2 - a^2}$$

$$6.23.- \int \frac{dx}{x\sqrt{x^2 - 9}}$$

$$6.26.- \int \frac{x^2 dx}{\sqrt{1 - x^2}}$$

$$6.29.- \int \frac{dx}{x\sqrt{4x^2 - 16}}$$

$$6.32.- \int \sqrt{a - x^2} dx$$

$$6.35.- \int \frac{dx}{x^2 \sqrt{x^2 + 9}}$$

$$6.38.- \int x^2 \sqrt{5 - x^2} dx$$

$$6.41.- \int \frac{dx}{x^2 \sqrt{x^2 + a^2}}$$

$$6.44.- \int \frac{dx}{x^2 \sqrt{a^2 - x^2}}$$

$$6.47.- \int \frac{\sqrt{x^2 - 100}}{x} dx$$

$$6.50.- \int \frac{\sqrt{x^2 + a^2}}{x} dx$$

$$6.53.- \int \frac{dx}{\sqrt{4 + x^2}}$$

$$6.56.- \int \frac{(x+1)dx}{\sqrt{4 - x^2}}$$

$$6.59.- \int \frac{dx}{\sqrt{4 - (x-1)^2}}$$

$$6.62.- \int \frac{x^2 dx}{\sqrt{21 + 4x - x^2}}$$

$$6.65.- \int \frac{dx}{(x-1)\sqrt{x^2 - 3x + 2}}$$

$$6.68.- \int \frac{(x-1)dx}{\sqrt{x^2 - 4x + 3}}$$

$$6.21.- \int \frac{dx}{\sqrt{x^2 + a^2}}$$

$$6.24.- \int \frac{dx}{x\sqrt{x^2 - 2}}$$

$$6.27.- \int \frac{x^3 dx}{\sqrt{2 - x^2}}$$

$$6.30.- \int \frac{\sqrt{x^2 + 1}}{x} dx$$

$$6.33.- \int \sqrt{a^2 - x^2} dx$$

$$6.36.- \int \frac{dx}{\sqrt{5 - 4x^2}}$$

$$6.39.- \int \frac{dx}{x^4 \sqrt{x^2 + 3}}$$

$$6.42.- \int \frac{dx}{(x^2 + a^2)^2}$$

$$6.45.- \int \frac{\sqrt{2x^2 - 5}}{x} dx$$

$$6.48.- \int \frac{dx}{x^2 \sqrt{x^2 - 2}}$$

$$6.51.- \int \frac{xdx}{\sqrt{a^2 - x^2}}$$

$$6.54.- \int \frac{xdx}{\sqrt{4 + x^2}}$$

$$6.57.- \int \frac{dx}{\sqrt{2 - 5x^2}}$$

$$6.60.- \int \frac{x^2 dx}{\sqrt{2x - x^2}}$$

$$6.63.- \int \frac{dx}{(x^2 - 2x + 5)^{3/2}}$$

$$6.66.- \int \frac{xdx}{\sqrt{x^2 - 2x + 5}}$$

$$6.69.- \int \frac{dx}{\sqrt{x^2 - 2x - 8}}$$

$$6.22.- \int \frac{dx}{\sqrt{x^2 - a^2}}$$

$$6.25.- \int \frac{dx}{x\sqrt{1 + x^2}}$$

$$6.28.- \int \frac{\sqrt{x^2 - 9}}{x} dx$$

$$6.31.- \int \frac{dx}{x^2 \sqrt{4 - x^2}}$$

$$6.34.- \int \frac{x^2 dx}{\sqrt{x^2 + a^2}}$$

$$6.37.- \int \frac{x^2 dx}{(4 - x^2)^{3/2}}$$

$$6.40.- \int x^3 \sqrt{a^2 x^2 + b^2} dx$$

$$6.43.- \int x^3 \sqrt{a^2 x^2 - b^2} dx$$

$$6.46.- \int \frac{x^3 dx}{\sqrt{3x^2 - 5}}$$

$$6.49.- \int \frac{dx}{x\sqrt{9 - x^2}}$$

$$6.52.- \int \frac{dx}{\sqrt{1 - 4x^2}}$$

$$6.55.- \int \frac{dx}{x\sqrt{a^2 + x^2}}$$

$$6.58.- \int \frac{dx}{(a^2 - x^2)^{3/2}}$$

$$6.61.- \int \frac{x^2 dx}{\sqrt{17 - x^2}}$$

$$6.64.- \int \frac{(2x+1)dx}{\sqrt{(4x^2 - 2x + 1)^3}}$$

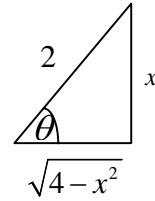
$$6.67.- \int \frac{(x+1)dx}{\sqrt{2x - x^2}}$$

$$6.70.- \int \frac{xdx}{\sqrt{x^2 + 4x + 5}}$$

RESPUESTAS

6.17.- $\int \sqrt{4-x^2}$

Solución.-



Se tiene: $x = 2 \operatorname{sen} \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta$

$$\int \sqrt{4-x^2} = \int 2 \cos \theta 2 \cos \theta d\theta = 4 \int \cos^2 \theta d\theta = 2\theta + \operatorname{sen} 2\theta + c = 2\theta + 2 \operatorname{sen} \theta \cos \theta + c$$

$$= 2 \operatorname{arcsen} \frac{x}{2} + \frac{x\sqrt{4-x^2}}{2} + c$$

6.18.- $\int \frac{dx}{\sqrt{a^2-x^2}}$

Solución.- se tiene: $x = a \operatorname{sen} \theta, dx = a \cos \theta d\theta, \sqrt{a^2-x^2} = a \cos \theta$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a \cos \theta d\theta}{a \cos \theta} = \int d\theta = \theta + c = \operatorname{arcsen} \frac{x}{a} + c$$

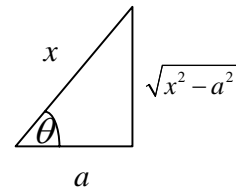
6.19.- $\int \frac{dx}{x^2+a^2}$

Solución.- se tiene: $x = a \operatorname{tg} \theta, dx = a \operatorname{sec}^2 \theta d\theta, \sqrt{x^2+a^2} = a \operatorname{sec} \theta$

$$\int \frac{dx}{x^2+a^2} = \int \frac{dx}{(\sqrt{x^2+a^2})^2} = \int \frac{a \operatorname{sec}^2 \theta d\theta}{a^2 \operatorname{sec}^2 \theta} = \frac{1}{a} \int d\theta = \frac{1}{a} \theta + c = \frac{1}{a} \operatorname{arc} \operatorname{tg} \frac{x}{a} + c$$

6.20.- $\int \frac{dx}{x^2-a^2}$

Solución.-



Se tiene: $x = a \operatorname{sec} \theta, dx = a \operatorname{sec} \theta \operatorname{tg} \theta d\theta, \sqrt{x^2-a^2} = a \operatorname{tg} \theta$

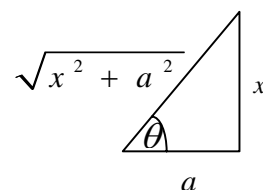
$$\int \frac{dx}{x^2-a^2} = \int \frac{dx}{(\sqrt{x^2-a^2})^2} = \int \frac{a \operatorname{sec} \theta \operatorname{tg} \theta d\theta}{a^2 \operatorname{tg}^2 \theta} = \frac{1}{a} \int \frac{\operatorname{sec} \theta d\theta}{\operatorname{tg} \theta} = \frac{1}{a} \int \operatorname{cosec} \theta d\theta$$

$$= \frac{1}{a} \ell \eta |\operatorname{cosec} \theta - \operatorname{co} \operatorname{tg} \theta| = \frac{1}{a} \ell \eta \left| \frac{x}{\sqrt{x^2-a^2}} - \frac{a}{\sqrt{x^2-a^2}} \right| + c$$

$$= \frac{1}{a} \ell \eta \left| \frac{x-a}{\sqrt{x^2-a^2}} \right| + c = \frac{1}{a} \ell \eta \left| \sqrt{\frac{(x-a)^2}{x^2-a^2}} \right| + c = \frac{1}{2a} \ell \eta \left| \frac{x-a}{x+a} \right| + c$$

6.21.- $\int \frac{dx}{\sqrt{x^2+a^2}}$

Solución.-

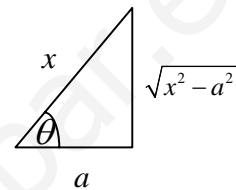


Se tiene: $x = a \tau g \theta, dx = a \sec^2 \theta d\theta, \sqrt{x^2 + a^2} = a \sec \theta$

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2 + a^2}} &= \int \frac{\cancel{a} \sec^2 \theta d\theta}{\cancel{a} \sec \theta} = \int \sec \theta d\theta = \ell \eta |\sec \theta + \tau g \theta| + c \\ &= \ell \eta \left| \frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right| + c = \ell \eta \left| \frac{\sqrt{x^2 + a^2} + x}{a} \right| + c = \ell \eta |x + \sqrt{x^2 + a^2}| - \ell \eta a + c \\ &= \ell \eta |x + \sqrt{x^2 + a^2}| + c \end{aligned}$$

6.22.- $\int \frac{dx}{\sqrt{x^2 - a^2}}$

Solución.-



Se tiene: $x = a \sec \theta, dx = a \sec \theta \tau g \theta d\theta, \sqrt{x^2 - a^2} = a \tau g \theta$

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2 - a^2}} &= \int \frac{\cancel{a} \sec \theta \tau g \theta d\theta}{\cancel{a} \tau g \theta} = \int \sec \theta d\theta = \ell \eta |\sec \theta + \tau g \theta| + c \\ &= \ell \eta \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + c = \ell \eta \left| \frac{x + \sqrt{x^2 - a^2}}{a} \right| + c = \ell \eta |x + \sqrt{x^2 - a^2}| + c \end{aligned}$$

6.23.- $\int \frac{dx}{x\sqrt{x^2 - 9}}$

Solución.-

Se tiene: $x = 3 \sec \theta, dx = 3 \sec \theta \tau g \theta d\theta, \sqrt{x^2 - 9} = 3 \tau g \theta$

$$\int \frac{dx}{x\sqrt{x^2 - 9}} = \int \frac{\cancel{3} \sec \theta \tau g \theta d\theta}{\cancel{3} \sec \theta \cancel{3} \tau g \theta} = \frac{1}{3} \int d\theta = \frac{1}{3} \theta + c = \frac{\text{arc sec } x/3}{3} + c$$

6.24.- $\int \frac{dx}{x\sqrt{x^2 - 2}}$

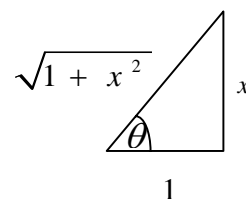
Solución.-

Se tiene: $x = \sqrt{2} \sec \theta, dx = \sqrt{2} \sec \theta \tau g \theta d\theta, \sqrt{x^2 - 2} = \sqrt{2} \tau g \theta$

$$\int \frac{dx}{x\sqrt{x^2 - 2}} = \int \frac{\cancel{\sqrt{2}} \sec \theta \tau g \theta d\theta}{\cancel{\sqrt{2}} \sec \theta \cancel{\sqrt{2}} \tau g \theta} = \frac{\sqrt{2}}{2} \int d\theta = \frac{\sqrt{2}}{2} \theta + c = \frac{\sqrt{2}}{2} \text{arc sec } \frac{\sqrt{2}}{2} x + c$$

6.25.- $\int \frac{dx}{x\sqrt{1+x^2}}$

Solución.-



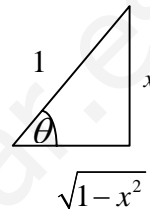
Se tiene: $x = \tau g \theta, dx = \sec^2 \theta d\theta, \sqrt{1+x^2} = \sec \theta$

$$\int \frac{dx}{x\sqrt{1+x^2}} = \int \frac{\sec^2 \theta d\theta}{\tau g \theta \sec \theta} = \int \frac{d\theta}{\sec \theta} = \int \cos \theta d\theta = \ell \eta |\cos \theta - \cot \theta| + c$$

$$= \ell \eta \left| \frac{\sqrt{1+x^2}}{x} - \frac{1}{x} \right| + c = \ell \eta \left| \frac{\sqrt{1+x^2} - 1}{x} \right| + c$$

6.26.- $\int \frac{x^2 dx}{\sqrt{1-x^2}}$

Solución.-



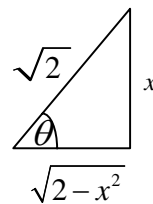
Se tiene: $x = \text{sen } \theta, dx = \cos \theta d\theta, \sqrt{1-x^2} = \cos \theta$

$$\int \frac{x^2 dx}{\sqrt{1-x^2}} = \int \frac{\text{sen}^2 \theta \cos \theta d\theta}{\cos \theta} = \int \text{sen}^2 \theta d\theta = \frac{1}{2} \theta - \frac{1}{4} \text{sen } 2\theta + c$$

$$= \frac{1}{2} \theta - \frac{1}{2} \text{sen } \theta \cos \theta + c = \frac{\arcsen x}{2} - \frac{x}{2} \sqrt{1-x^2} + c$$

6.27.- $\int \frac{x^3 dx}{\sqrt{2-x^2}}$

Solución.-



Se tiene: $x = \sqrt{2} \text{sen } \theta, dx = \sqrt{2} \cos \theta d\theta, \sqrt{2-x^2} = \sqrt{2} \cos \theta$

$$\int \frac{x^3 dx}{\sqrt{2-x^2}} = \int \frac{2\sqrt{2} \text{sen}^3 \theta \sqrt{2} \cos \theta d\theta}{\sqrt{2} \cos \theta} = 2\sqrt{2} \int \text{sen}^3 \theta d\theta = 2\sqrt{2} \left(-\cos \theta + \frac{\cos^3 \theta}{3} \right) + c$$

$$= 2\sqrt{2} \left(-\frac{\sqrt{2-x^2}}{\sqrt{2}} + \frac{(\sqrt{2-x^2})^3}{3(\sqrt{2})^3} \right) + c = -\sqrt{2(2-x^2)} + \frac{(2-x^2)\sqrt{2-x^2}}{3} + c$$

6.28.- $\int \frac{\sqrt{x^2-9}}{x} dx$

Solución.-

Se tiene: $x = 3 \sec \theta, dx = 3 \sec \theta \tau g \theta d\theta, \sqrt{x^2-9} = 3 \tau g \theta$

$$\int \frac{\sqrt{x^2-9}}{x} dx = \int \frac{3 \tau g \theta 3 \sec \theta \tau g \theta d\theta}{3 \sec \theta} = 3 \int \tau g^2 \theta d\theta = 3 \int (\sec^2 \theta - 1) d\theta$$

$$= 3 \int \sec^2 \theta d\theta - 3 \int d\theta = 3 \tau g \theta - 3\theta + c = \sqrt{x^2-9} - 3 \text{arc sec } \frac{x}{3} + c$$

$$6.29.- \int \frac{dx}{x\sqrt{4x^2-16}}$$

Solución.-

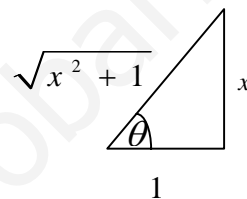
$$\text{Se tiene: } \frac{x}{2} = \sec \theta, dx = 2 \sec \theta \tau g \theta d\theta, \sqrt{\frac{x^2}{4}-1} = \tau g \theta$$

$$\int \frac{dx}{x\sqrt{4x^2-16}} = \frac{1}{4} \int \frac{dx}{x\sqrt{(x/2)^2-1}} = \frac{1}{4} \int \frac{2\sec\theta\tau g\theta d\theta}{2\sec\theta\tau g\theta} = \frac{1}{4} \int d\theta = \frac{1}{4}\theta + c$$

$$= \frac{1}{4} \operatorname{arc} \sec \frac{x}{2} + c$$

$$6.30.- \int \frac{\sqrt{x^2+1}}{x} dx$$

Solución.-



$$\text{Se tiene: } x = \tau g \theta, dx = \sec^2 \theta d\theta, \sqrt{x^2+1} = \sec \theta$$

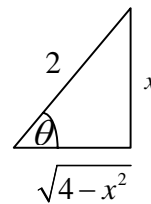
$$\int \frac{\sqrt{x^2+1}}{x} dx = \int \frac{\sec \theta \sec^2 \theta d\theta}{\tau g \theta} = \int \frac{d\theta}{\cos^2 \theta \operatorname{sen} \theta} = \ell \eta \left| \tau g \frac{\theta}{2} \right| + \frac{1}{\cos \theta} + c, \text{ o bien:}$$

$$= \ell \eta \left| \operatorname{cosec} \theta - \operatorname{co} \tau g \theta \right| + \frac{1}{\cos \theta} + c = \ell \eta \left| \frac{\sqrt{x^2+1}}{x} - \frac{1}{x} \right| + \frac{1}{\sqrt{x^2+1}} + c$$

$$= \ell \eta \left| \frac{\sqrt{x^2+1}-1}{x} \right| + \sqrt{x^2+1} + c$$

$$6.31.- \int \frac{dx}{x^2\sqrt{4-x^2}}$$

Solución.-



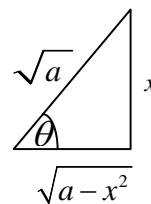
$$\text{Se tiene: } x = 2 \operatorname{sen} \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta$$

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta d\theta}{4\operatorname{sen}^2\theta 2\cos\theta} = \frac{1}{4} \int \operatorname{cosec}^2 \theta d\theta = -\frac{1}{4} \operatorname{co} \tau g \theta + c$$

$$= -\frac{\sqrt{4-x^2}}{4x} + c$$

$$6.32.- \int \sqrt{a-x^2} dx$$

Solución.-



Se tiene: $x = \sqrt{a} \operatorname{sen} \theta, dx = \sqrt{a} \cos \theta d\theta, \sqrt{a-x^2} = \sqrt{a} \cos \theta$

$$\int \sqrt{a-x^2} dx = \int \sqrt{a} \cos \theta \sqrt{a} \cos \theta d\theta = a \int \cos^2 \theta d\theta$$

$$\frac{a}{2} \theta + \frac{a}{2} \operatorname{sen} \theta \cos \theta + c = \frac{a}{2} \operatorname{arcsen} \frac{x}{\sqrt{a}} + \frac{x}{2} \sqrt{a^2 - x^2} + c$$

6.33.- $\int \sqrt{a^2 - x^2} dx$

Solución.-

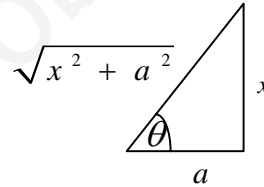
Se tiene: $x = a \operatorname{sen} \theta, dx = a \cos \theta d\theta, \sqrt{a^2 - x^2} = a \cos \theta$

$$\int \sqrt{a^2 - x^2} dx = \int a \cos \theta a \cos \theta d\theta = a^2 \int \cos^2 \theta d\theta$$

$$\frac{a^2}{2} \theta + \frac{a^2}{2} \operatorname{sen} \theta \cos \theta + c = \frac{a^2}{2} \operatorname{arcsen} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + c$$

6.34.- $\int \frac{x^2 dx}{\sqrt{x^2 + a^2}}$

Solución.-



Se tiene: $x = a \operatorname{tg} \theta, dx = a \sec^2 \theta d\theta, \sqrt{x^2 + a^2} = a \sec \theta$

$$\int \frac{x^2 dx}{\sqrt{x^2 + a^2}} = \int \frac{a^2 \operatorname{tg}^2 \theta \cancel{a} \sec^2 \theta d\theta}{a \sec \theta} = a^2 \int \operatorname{tg}^2 \theta \sec \theta d\theta = a^2 \int \frac{\operatorname{sen}^2 \theta}{\cos^3 \theta} d\theta$$

$$= a^2 \int \frac{(1 - \cos^2 \theta)}{\cos^3 \theta} d\theta = a^2 \int \sec^3 \theta d\theta - a^2 \int \sec \theta d\theta$$

$$= a^2 \left(\frac{\sec \theta \operatorname{tg} \theta}{2} + \frac{1}{2} \ell \eta |\sec \theta + \operatorname{tg} \theta| \right) - a^2 \ell \eta |\sec \theta + \operatorname{tg} \theta| + c$$

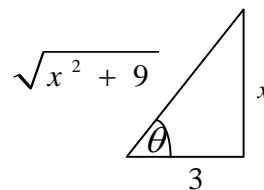
$$= \frac{a^2}{2} \sec \theta \operatorname{tg} \theta + \frac{a^2}{2} \ell \eta |\sec \theta + \operatorname{tg} \theta| - a^2 \ell \eta |\sec \theta + \operatorname{tg} \theta| + c$$

$$= \frac{a^2}{2} \sec \theta \operatorname{tg} \theta - \frac{a^2}{2} \ell \eta |\sec \theta + \operatorname{tg} \theta| + c$$

$$= \frac{a^2}{2} \frac{\sqrt{x^2 + a^2}}{a} \frac{x}{a} - \frac{a^2}{2} \ell \eta \left| \frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right| + c = \frac{x \sqrt{x^2 + a^2}}{2} - \frac{a^2}{2} \ell \eta \left| \sqrt{x^2 + a^2} + x \right| + c$$

6.35.- $\int \frac{dx}{x^2 \sqrt{x^2 + 9}}$

Solución.-



Se tiene: $x = 3 \tau g \theta, dx = 3 \sec^2 \theta d\theta, \sqrt{x^2 + 9} = 3 \sec \theta$

$$\int \frac{dx}{x^2 \sqrt{x^2 + 9}} = \int \frac{\cancel{3} \sec^2 \theta d\theta}{9 \tau g^2 \theta \cancel{3 \sec \theta}} = \frac{1}{9} \int \frac{\sec \theta d\theta}{\tau g^2 \theta} = \frac{1}{9} \int \frac{\cos \theta}{\text{sen}^2 \theta} d\theta = -\frac{1}{9 \text{sen} \theta} + c$$

$$= -\frac{\sqrt{x^2 + 9}}{9x} + c$$

6.36.- $\int \frac{dx}{\sqrt{5-4x^2}}$

Solución.-

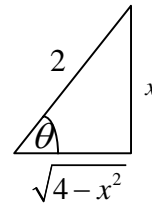
Se tiene: $x = \sqrt{5/4} \text{sen} \theta, dx = \sqrt{5/4} \cos \theta d\theta, \sqrt{(5/4)^2 - x^2} = 5/4 \cos \theta$

$$\int \frac{dx}{\sqrt{5-4x^2}} = \frac{1}{2} \int \frac{dx}{\sqrt{5/4 - x^2}} = \frac{1}{2} \int \frac{\cancel{\sqrt{5/4}} \cos \theta d\theta}{\sqrt{5/4} \cancel{\cos \theta}} = \frac{1}{2} \int d\theta = \frac{1}{2} \theta + c$$

$$= \frac{1}{2} \arcsen \frac{x}{\sqrt{5/4}} + c = \frac{1}{2} \arcsen \frac{2x}{\sqrt{5}} + c$$

6.37.- $\int \frac{x^2 dx}{(4-x^2)^{3/2}}$

Solución.-



Se tiene: $x = 2 \text{sen} \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta$

$$\int \frac{x^2 dx}{(4-x^2)^{3/2}} = \int \frac{x^2 dx}{\sqrt{(4-x^2)^3}} = \int \frac{\cancel{4} \text{sen}^2 \theta \cancel{2} \cos \theta d\theta}{\cancel{8} \cos^3 \theta} = \int \tau g^2 \theta d\theta = \int (\sec^2 \theta - 1) d\theta$$

$$= \tau g \theta - \theta + c = \frac{x}{\sqrt{4-x^2}} - \arcsen \frac{x}{2} + c$$

6.38.- $\int x^2 \sqrt{5-x^2} dx$

Solución.-

Se tiene: $x = \sqrt{5} \text{sen} \theta, dx = \sqrt{5} \cos \theta d\theta, \sqrt{5-x^2} = \sqrt{5} \cos \theta$

$$\int x^2 \sqrt{5-x^2} dx = \int 5 \text{sen}^2 \theta \sqrt{5} \cos \theta \sqrt{5} \cos \theta d\theta = 25 \int \text{sen}^2 \theta \cos^2 \theta d\theta = \frac{25}{4} \int \text{sen}^2 2\theta d\theta$$

$$= \frac{25}{8} \int (1 - \cos 4\theta) d\theta = \frac{25}{8} \theta - \frac{25}{32} \text{sen} 4\theta + c = \frac{25}{8} \theta - \frac{25}{32} (2 \text{sen} 2\theta \cos 2\theta) + c$$

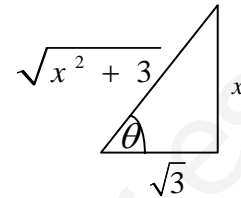
$$= \frac{25}{8} \theta - \frac{25}{32} [2 \text{sen} \theta \cos 2\theta (\cos^2 \theta - \text{sen}^2 \theta)] + c$$

$$= \frac{25}{8}\theta - \frac{25}{16}[\operatorname{sen}\theta \cos^3\theta - \operatorname{sen}^3\theta \cos\theta] + c$$

$$= \frac{25}{2} \left[\operatorname{arcsen} \frac{x}{\sqrt{5}} - \frac{x(\sqrt{5-x^2})^3}{25} + \frac{x^3\sqrt{5-x^2}}{25} \right] + c$$

6.39.- $\int \frac{dx}{x^4\sqrt{x^2+3}}$

Solución.-



Se tiene: $x = \sqrt{3}\tau g\theta$, $dx = \sqrt{3}\sec^2\theta d\theta$, $\sqrt{x^2+3} = \sqrt{3}\sec\theta$

$$\int \frac{dx}{x^4\sqrt{x^2+3}} = \int \frac{\sqrt{3}\sec^2\theta d\theta}{9\tau g^4\theta \sqrt{3}\sec\theta} = \frac{1}{9} \int \frac{\sec\theta d\theta}{\tau g^4\theta} = \frac{1}{9} \int \frac{\cos^3\theta d\theta}{\operatorname{sen}^4\theta} = \frac{1}{9} \int \frac{(1-\operatorname{sen}^2\theta)\cos\theta d\theta}{\operatorname{sen}^4\theta}$$

$$= \frac{1}{9} \int \frac{\cos\theta d\theta}{\operatorname{sen}^4\theta} - \frac{1}{9} \int \frac{\cos\theta d\theta}{\operatorname{sen}^2\theta} = -\frac{1}{27}\operatorname{cosec}^3\theta + \frac{1}{9}\operatorname{cosec}\theta + c = \frac{\sqrt{x^2+3}}{9x} - \left(\frac{\sqrt{x^2+3}}{3x}\right)^3 + c$$

6.40.- $\int x^3\sqrt{a^2x^2+b^2} dx$

Solución.-

Se tiene: $ax = b\tau g\theta$, $adx = b\sec^2\theta d\theta$, $\sqrt{a^2x^2+b^2} = b\sec\theta$

$$\int x^3\sqrt{a^2x^2+b^2} dx = \int \frac{b^3}{a^3}\tau g^3\theta b\sec\theta \frac{b}{a}\sec^2\theta d\theta = \frac{b^5}{a^4} \int \tau g^3\theta \sec^3\theta d\theta$$

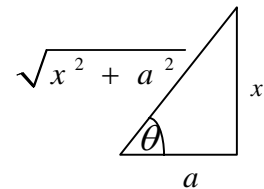
$$= \frac{b^5}{a^4} \int \tau g^2\theta \sec^2\theta \tau g\theta \sec\theta d\theta = \frac{b^5}{a^4} \int (\sec^2\theta - 1)\sec^2\theta \tau g\theta \sec\theta d\theta$$

$$= \frac{b^5}{a^4} \int \sec^4\theta \tau g\theta \sec\theta d\theta - \frac{b^5}{a^4} \int \sec^2\theta \tau g\theta \sec\theta d\theta = \frac{b^5}{a^4} \frac{\sec^5\theta}{5} + \frac{b^5}{a^4} \frac{\sec^3\theta}{3} + c$$

$$= \frac{b^5}{a^4} \left[\frac{(\sqrt{a^2x^2+b^2})^5}{5b^5} + \frac{(\sqrt{a^2x^2+b^2})^3}{3b^3} \right] + c = \frac{(a^2x^2+b^2)^{5/2}}{5a^4} - \frac{(a^2x^2+b^2)^{3/2}b^2}{3a^4} + c$$

6.41.- $\int \frac{dx}{x^2\sqrt{x^2+a^2}}$

Solución.-



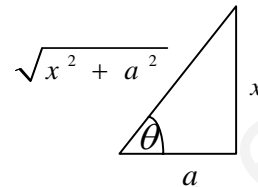
Se tiene: $x = a\tau g\theta$, $dx = a\sec^2\theta d\theta$, $\sqrt{x^2+a^2} = a\sec\theta$

$$\int \frac{dx}{x^2 \sqrt{x^2 + a^2}} = \int \frac{a \sec^2 \theta d\theta}{a^2 \tau g^2 \theta a \sec \theta} = \frac{1}{a^2} \int \frac{\sec \theta d\theta}{\tau g^2 \theta} = \frac{1}{a^2} \int \frac{\cos \theta d\theta}{\text{sen}^2 \theta} d\theta$$

$$= \frac{1}{a^2} \int \cot \tau g \theta \cos \text{ec} \theta d\theta = -\frac{\cos \text{ec} \theta}{a^2} + c = -\frac{1}{a^2 x} \sqrt{x^2 + a^2} + c$$

6.42.- $\int \frac{dx}{(x^2 + a^2)^2}$

Solución.-



Se tiene: $x = a \tau g \theta, dx = a \sec^2 \theta d\theta, \sqrt{x^2 + a^2} = a \sec \theta$

$$\int \frac{dx}{(x^2 + a^2)^2} = \int \frac{dx}{(\sqrt{x^2 + a^2})^4} = \int \frac{a \sec^2 \theta d\theta}{a^4 \sec^4 \theta} = \frac{1}{a^3} \int \cos^2 \theta d\theta = \frac{1}{2a^3} \theta + \frac{1}{2a^3} \frac{\text{sen} 2\theta}{2} + c$$

$$= \frac{1}{2a^3} \theta + \frac{1}{2a^3} \frac{\text{sen} \theta \cos \theta}{2} + c = \frac{1}{2a^3} \text{arc} \tau g \frac{x}{a} + \frac{1}{2a^3} \left(\frac{x}{\sqrt{x^2 + a^2}} \frac{a}{\sqrt{x^2 + a^2}} \right) + c$$

$$= \frac{1}{2a^3} \text{arc} \tau g \frac{x}{a} + \frac{1}{2a^3} \left(\frac{ax}{\sqrt{x^2 + a^2}} \right) + c$$

6.43.- $\int x^3 \sqrt{a^2 x^2 - b^2} dx$

Solución.-

Se tiene: $ax = b \sec \theta, adx = b \sec \theta \tau g \theta d\theta, \sqrt{a^2 x^2 - b^2} = b \tau g \theta$

$$\int x^3 \sqrt{a^2 x^2 - b^2} dx = \int \frac{b^3}{a^3} \sec^3 \theta b \tau g \theta \frac{b}{a} \sec \theta \tau g \theta d\theta = \frac{b^5}{a^4} \int \sec^4 \theta \tau g^2 \theta d\theta$$

$$= \frac{b^5}{a^4} \int \sec^4 \theta (\sec^2 \theta - 1) d\theta = \frac{b^5}{a^4} \int \sec^4 \theta \sec^2 \theta d\theta - \frac{b^5}{a^4} \int \sec^2 \theta \sec^2 \theta d\theta$$

$$= \frac{b^5}{a^4} \int (1 + \tau g^2 \theta)^2 \sec^2 \theta d\theta - \frac{b^5}{a^4} \int (1 + \tau g^2 \theta) \sec^2 \theta d\theta$$

$$= \frac{b^5}{a^4} \int (1 + 2\tau g^2 \theta + \tau g^4 \theta) \sec^2 \theta d\theta - \frac{b^5}{a^4} \int (1 + \tau g^2 \theta) \sec^2 \theta d\theta$$

$$= \frac{b^5}{a^4} \left[\int \tau g^2 \theta \sec^2 \theta d\theta + \int \tau g^4 \theta \sec^2 \theta d\theta \right] = \frac{b^5}{a^4} \left[\frac{\tau g^3 \theta}{3} + \frac{\tau g^5 \theta}{5} \right] + c$$

$$= \frac{b^5}{a^4} \left[\frac{1}{3} \left(\frac{\sqrt{a^2 x^2 - b^2}}{b} \right)^3 + \frac{1}{5} \left(\frac{\sqrt{a^2 x^2 - b^2}}{b} \right)^5 \right] + c$$

6.44.- $\int \frac{dx}{x^2 \sqrt{a^2 - x^2}}$

Solución.-

Se tiene: $x = a \operatorname{sen} \theta, dx = a \cos \theta d\theta, \sqrt{a^2 - x^2} = a \cos \theta$

$$\int \frac{dx}{x^2 \sqrt{a^2 - x^2}} = \int \frac{a \cancel{\cos \theta} d\theta}{a^2 \operatorname{sen}^2 \theta \cancel{a \cos \theta}} = \frac{1}{a^2} \int \cos \theta c^2 \theta d\theta = -\frac{1}{a^2} \cot \theta + c$$

$$= -\frac{1}{a^2} \frac{\cos \theta}{\operatorname{sen} \theta} + c = -\frac{1}{a^2} \left(\frac{\sqrt{a^2 - x^2}}{x} \right) + c$$

6.45.- $\int \frac{\sqrt{2x^2 - 5}}{x} dx$

Solución.-

Se tiene: $\sqrt{2}x = \sqrt{5} \sec \theta, \sqrt{2}dx = \sqrt{5} \sec \theta \tau g \theta d\theta, \sqrt{2x^2 - 5} = \sqrt{5} \tau g \theta$

$$\int \frac{\sqrt{2x^2 - 5}}{x} dx = \int \frac{\sqrt{5} \tau g \theta \cancel{\sqrt{5}} \sec \theta \tau g \theta d\theta}{\cancel{\sqrt{2}} \sec \theta} = \sqrt{5} \int \tau g^2 \theta d\theta = \sqrt{5} \int \sec^2 \theta d\theta - \sqrt{5} \int d\theta$$

$$= \sqrt{5} \tau g \theta - \sqrt{5} \theta + c = \sqrt{2x^2 - 5} - \sqrt{5} \operatorname{arc} \sec \sqrt{\frac{2}{3}} x + c$$

6.46.- $\int \frac{x^3 dx}{\sqrt{3x^2 - 5}}$

Solución.-

Se tiene: $\sqrt{3}x = \sqrt{5} \sec \theta, \sqrt{3}dx = \sqrt{5} \sec \theta \tau g \theta d\theta, \sqrt{3x^2 - 5} = \sqrt{5} \tau g \theta$

$$\int \frac{x^3 dx}{\sqrt{3x^2 - 5}} = \int \frac{(\sqrt{5/3} \sec \theta)^3 \sqrt{5/3} \sec \theta \tau g \theta d\theta}{\sqrt{5/3} \tau g \theta} = \frac{5\sqrt{5}}{9} \int \sec^4 \theta d\theta$$

$$= \frac{5\sqrt{5}}{9} \int \sec^2 \theta \sec^2 \theta d\theta = \frac{5\sqrt{5}}{9} \int \sec^2 \theta (1 + \tau g^2 \theta) d\theta$$

$$= \frac{5\sqrt{5}}{9} \left[\int \sec^2 \theta d\theta + \int \sec^2 \theta \tau g^2 \theta d\theta \right] = \frac{5\sqrt{5}}{9} \left[\tau g \theta + \frac{\tau g^3 \theta}{3} \right] + c$$

$$= \frac{5}{9} \left[\sqrt{3x^2 - 5} + \frac{(\sqrt{3x^2 - 5})^3}{15} \right] + c$$

6.47.- $\int \frac{\sqrt{x^2 - 100}}{x} dx$

Solución.-

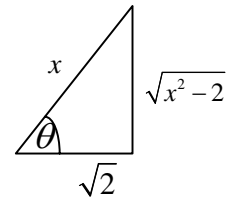
Se tiene: $x = 10 \sec \theta, dx = 10 \sec \theta \tau g \theta d\theta, \sqrt{x^2 - 100} = 10 \tau g \theta$

$$\int \frac{\sqrt{x^2 - 100}}{x} dx = \int \frac{10 \tau g \theta \cancel{10 \sec \theta} \tau g \theta d\theta}{10 \sec \theta} = 10 \int \tau g^2 \theta d\theta = 10 \int \sec^2 \theta - 10 \int d\theta$$

$$= 10(\tau g \theta - \theta) + c = \sqrt{x^2 - 100} - 10 \operatorname{arcs} \operatorname{en} \frac{x}{10} + c$$

$$6.48.- \int \frac{dx}{x^2 \sqrt{x^2 - 2}}$$

Solución.-

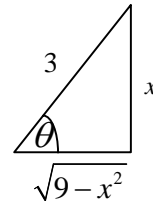


Se tiene: $x = \sqrt{2} \sec \theta, dx = \sqrt{2} \sec \theta \tan \theta d\theta, \sqrt{x^2 - 2} = \sqrt{2} \tan \theta$

$$\begin{aligned} \int \frac{dx}{x^2 \sqrt{x^2 - 2}} &= \int \frac{\sqrt{2} \sec \theta \tan \theta d\theta}{2 \sec^2 \theta \sqrt{2} \tan \theta} = \frac{1}{2} \int \cos \theta d\theta = \frac{1}{2} \operatorname{sen} \theta + c = \frac{1}{2} \frac{\sqrt{x^2 - 2}}{x} + c \\ &= \frac{\sqrt{x^2 - 2}}{2x} + c \end{aligned}$$

$$6.49.- \int \frac{dx}{x \sqrt{9 - x^2}}$$

Solución.-

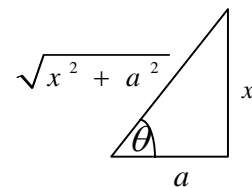


Se tiene: $x = 3 \operatorname{sen} \theta, dx = 3 \cos \theta d\theta, \sqrt{9 - x^2} = 3 \cos \theta$

$$\begin{aligned} \int \frac{dx}{x \sqrt{9 - x^2}} &= \int \frac{3 \cos \theta d\theta}{3 \operatorname{sen} \theta 3 \cos \theta} = \frac{1}{3} \int \operatorname{cosec} \theta d\theta = \frac{1}{3} \ln |\operatorname{cosec} \theta - \cot \theta| + c \\ &= \frac{1}{3} \ln \left| \frac{3 - \sqrt{9 - x^2}}{x} \right| + c \end{aligned}$$

$$6.50.- \int \frac{\sqrt{x^2 + a^2}}{x} dx$$

Solución.-



Se tiene: $x = a \sec \theta, dx = a \sec^2 \theta d\theta, \sqrt{x^2 + a^2} = a \sec \theta$

$$\begin{aligned} \int \frac{\sqrt{x^2 + a^2}}{x} dx &= \int \frac{a \sec \theta}{a \sec \theta} \sec^2 \theta d\theta = a \int \frac{\sec^3 \theta d\theta}{\sec \theta} = a \int \frac{\sec^2 \theta \sec \theta}{\sec \theta} d\theta \\ &= a \int \frac{(1 + \tan^2 \theta) \sec \theta}{\sec \theta} d\theta = a \int \sec \theta d\theta + a \int \sec \theta \tan \theta d\theta \end{aligned}$$

$$a \ln |\operatorname{cosec} \theta - \cot \theta| + a \sec \theta + c = a \ln \left| \frac{\sqrt{x^2 + a^2} - a}{x} \right| + \sqrt{x^2 + a^2} + c$$

$$6.51.- \int \frac{xdx}{\sqrt{a^2 - x^2}}$$

Solución.-

Se tiene: $x = a \operatorname{sen} \theta, dx = a \cos \theta d\theta, \sqrt{a^2 - x^2} = a \cos \theta$

$$\int \frac{xdx}{\sqrt{a^2 - x^2}} = \int \frac{a \operatorname{sen} \theta \cancel{a \cos \theta} d\theta}{a \cos \theta} = a \int \operatorname{sen} \theta d\theta = -a \cos \theta + c = -\sqrt{a^2 - x^2} + c$$

$$6.52.- \int \frac{dx}{\sqrt{1-4x^2}}$$

Solución.-

Se tiene: $2x = \operatorname{sen} \theta, 2dx = \cos \theta d\theta, \sqrt{1-4x^2} = \cos \theta$

$$\int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2} \int \frac{\cancel{\cos \theta} d\theta}{\cos \theta} = \frac{1}{2} \int d\theta = \frac{1}{2} \theta + c = \frac{1}{2} \operatorname{arcsen} 2x + c$$

$$6.53.- \int \frac{dx}{\sqrt{4+x^2}}$$

Solución.-

Se tiene: $x = 2 \operatorname{tg} \theta, dx = 2 \sec^2 \theta d\theta, \sqrt{4+x^2} = 2 \sec \theta$

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{\cancel{2} \sec^2 \theta d\theta}{2 \sec \theta} = \int \sec \theta d\theta = \ell \eta |\sec \theta + \operatorname{tg} \theta| + c = \ell \eta |\sqrt{4+x^2} + x| + c$$

$$6.54.- \int \frac{xdx}{\sqrt{4+x^2}}$$

Solución.-

Se tiene: $x = 2 \operatorname{tg} \theta, dx = 2 \sec^2 \theta d\theta, \sqrt{4+x^2} = 2 \sec \theta$

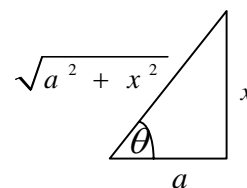
$$\int \frac{xdx}{\sqrt{4+x^2}} = \int \frac{2 \operatorname{tg} \theta \cancel{2} \sec^2 \theta d\theta}{2 \sec \theta} = 2 \int \operatorname{tg} \theta \sec \theta d\theta = 2 \sec \theta + c = \sqrt{4+x^2} + c$$

$$6.55.- \int \frac{dx}{x\sqrt{a^2+x^2}}$$

Solución.-

Se tiene: $x = a \operatorname{tg} \theta, dx = a \sec^2 \theta d\theta, \sqrt{a^2+x^2} = a \sec \theta$

$$\begin{aligned} \int \frac{dx}{x\sqrt{a^2+x^2}} &= \int \frac{\cancel{a} \sec^2 \theta d\theta}{a \operatorname{tg} \theta \cancel{a \sec \theta}} = \frac{1}{a} \int \frac{\sec \theta d\theta}{\operatorname{tg} \theta} = \frac{1}{a} \int \operatorname{cosec} \theta d\theta \\ &= \frac{1}{a} \ell \eta |\operatorname{cosec} \theta - \operatorname{co} \operatorname{tg} \theta| + c = \frac{1}{a} \ell \eta \left| \frac{\sqrt{a^2+x^2}}{x} - \frac{a}{x} \right| + c = \frac{1}{a} \ell \eta \left| \frac{\sqrt{a^2+x^2} - a}{x} \right| + c \end{aligned}$$



$$6.56.- \int \frac{(x+1)dx}{\sqrt{4-x^2}}$$

Solución.-

Se tiene: $x = 2 \operatorname{sen} \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta$

$$\int \frac{(x+1)dx}{\sqrt{4-x^2}} = \int \frac{x dx}{\sqrt{4-x^2}} + \int \frac{dx}{\sqrt{4-x^2}} = \int \frac{2 \operatorname{sen} \theta \cancel{2 \cos \theta} d\theta}{2 \cos \theta} + \int \frac{\cancel{2 \cos \theta} d\theta}{2 \cos \theta}$$

$$2 \int \operatorname{sen} \theta d\theta + \int d\theta = -2 \cos \theta + \theta + c = -\sqrt{4-x^2} + \operatorname{arcsen} \frac{x}{2} + c$$

6.57.- $\int \frac{dx}{\sqrt{2-5x^2}}$

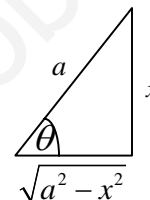
Solución.-

Se tiene: $\sqrt{5}x = \sqrt{2} \operatorname{sen} \theta, \sqrt{5}dx = \sqrt{2} \cos \theta d\theta, \sqrt{2-5x^2} = \sqrt{2} \cos \theta$

$$\int \frac{dx}{\sqrt{2-5x^2}} = \int \frac{\frac{\sqrt{2}}{\sqrt{5}} \cancel{\cos \theta} d\theta}{\sqrt{2} \cos \theta} = \frac{\sqrt{5}}{5} \int d\theta = \frac{\sqrt{5}}{5} \theta + c = \frac{\sqrt{5}}{5} \operatorname{arcsen} \sqrt{\frac{5}{2}} x + c$$

6.58.- $\int \frac{dx}{(a^2-x^2)^{3/2}}$

Solución.-



Se tiene: $x = a \operatorname{sen} \theta, dx = a \cos \theta d\theta, \sqrt{a^2-x^2} = a \cos \theta$

$$\int \frac{dx}{(a^2-x^2)^{3/2}} = \int \frac{dx}{(\sqrt{a^2-x^2})^3} = \int \frac{\cancel{a} \cancel{\cos \theta} d\theta}{a^3 \cos^3 \theta} = \frac{1}{a^2} \int \sec^2 \theta d\theta = \frac{1}{a^2} \operatorname{tg} \theta + c$$

$$= \frac{x}{a^2 \sqrt{a^2-x^2}} + c$$

6.59.- $\int \frac{dx}{\sqrt{4-(x-1)^2}}$

Solución.-

Se tiene: $x-1 = 2 \operatorname{sen} \theta, dx = 2 \cos \theta d\theta, \sqrt{4-(x-1)^2} = 2 \cos \theta$

$$\int \frac{dx}{\sqrt{4-(x-1)^2}} = \int \frac{\cancel{2 \cos \theta} d\theta}{2 \cos \theta} = \int d\theta = \theta + c = \operatorname{arcsen} \frac{x-1}{2} + c$$

6.60.- $\int \frac{x^2 dx}{\sqrt{2x-x^2}}$

Solución.-

Se tiene: $x-1 = \operatorname{sen} \theta \Rightarrow x = \operatorname{sen} \theta + 1, dx = \cos \theta d\theta, \sqrt{1-(x-1)^2} = \cos \theta$

Completando cuadrados se tiene:

$2x-x^2 = -(x^2-2x) = -(x^2-2x+1)+1 = 1-(x-1)^2$, luego:

$$\int \frac{x^2 dx}{\sqrt{2x-x^2}} = \int \frac{x^2 dx}{\sqrt{1-(x-1)^2}} = \int \frac{(\operatorname{sen} \theta + 1)^2 \cancel{\cos \theta} d\theta}{\cos \theta} = \int (\operatorname{sen} \theta + 1)^2 d\theta$$

$$\begin{aligned}
&= \int \sin^2 \theta d\theta + 2 \int \sin \theta d\theta + \int d\theta = \frac{1}{2} \int d\theta - \frac{1}{2} \int \cos 2\theta d\theta + 2 \int \sin \theta d\theta + \int d\theta \\
&= \frac{3}{2} \int d\theta - \frac{1}{2} \int \cos 2\theta d\theta + 2 \int \sin \theta d\theta = \frac{3}{2} \theta - \frac{1}{4} \sin 2\theta - 2 \cos \theta + c \\
&= \frac{3}{2} \theta - \frac{1}{2} \sin \theta \cos \theta - 2 \cos \theta + c = \frac{3}{2} \arcsen(x-1) - \frac{1}{2} (x-1) \sqrt{2x-x^2} - 2\sqrt{2x-x^2} + c
\end{aligned}$$

6.61.- $\int \frac{x^2 dx}{\sqrt{17-x^2}}$

Solución.-

Se tiene: $x = \sqrt{17} \sin \theta, dx = \sqrt{17} \cos \theta d\theta, \sqrt{17-x^2} = \sqrt{17} \cos \theta$

$$\begin{aligned}
\int \frac{x^2 dx}{\sqrt{17-x^2}} &= \int \frac{17 \sin^2 \theta \cancel{\sqrt{17} \cos \theta} d\theta}{\cancel{\sqrt{17} \cos \theta}} = 17 \int \sin^2 \theta d\theta = \frac{17}{2} \int d\theta - \frac{17}{2} \int \cos 2\theta d\theta \\
&= \frac{17}{2} \theta - \frac{17}{4} \sin 2\theta + c = \frac{17}{2} \theta - \frac{17}{2} \sin \theta \cos \theta + c \\
&= \frac{17}{2} \arcsen \frac{x}{\sqrt{17}} - \frac{17}{2} \frac{x}{\sqrt{17}} \frac{\sqrt{17-x^2}}{\sqrt{17}} + c = \frac{17}{2} \arcsen \frac{x}{\sqrt{17}} - \frac{1}{2} x \sqrt{17-x^2} + c
\end{aligned}$$

6.62.- $\int \frac{x^2 dx}{\sqrt{21+4x-x^2}}$

Solución.-

Se tiene: $x-2 = 5 \sin \theta \Rightarrow x = 5 \sin \theta + 2, dx = 5 \cos \theta d\theta, \sqrt{5^2-(x-2)^2} = 5 \cos \theta$

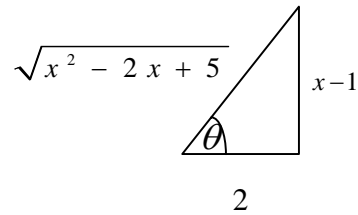
Completando cuadrados se tiene:

$21+4x-x^2 = -(x^2-4x+4-4)+21 = -(x^2-4x+4)+25 = 5^2-(x-2)^2$, luego:

$$\begin{aligned}
\int \frac{x^2 dx}{\sqrt{21+4x-x^2}} &= \int \frac{x^2 dx}{\sqrt{5^2-(x-2)^2}} = \int \frac{(5 \sin \theta + 2)^2 5 \cos \theta d\theta}{5 \cos \theta} = \int (5 \sin \theta + 2)^2 d\theta \\
&= \int (25 \sin^2 \theta + 20 \sin \theta + 4) d\theta = 25 \int \frac{1-\cos 2\theta}{2} d\theta + 20 \int \sin \theta d\theta + 4 \int d\theta \\
&= \frac{25}{2} \int d\theta - \frac{25}{2} \int \cos 2\theta d\theta + 20 \int \sin \theta d\theta = \frac{25}{2} \theta - \frac{25}{4} \sin 2\theta - 20 \cos \theta + 4\theta + c \\
&= \frac{33}{2} \theta - \frac{25}{2} \sin \theta \cos \theta - 20 \cos \theta + c \\
&= \frac{33}{2} \arcsen \frac{x-2}{5} - \frac{25}{2} \frac{x-2}{5} \left(\frac{\sqrt{21+4x-x^2}}{5} \right) - 20 \left(\frac{\sqrt{21+4x-x^2}}{5} \right) + c \\
&= \frac{33}{2} \arcsen \frac{x-2}{5} - \sqrt{21+4x-x^2} \left(\frac{x-2}{2} + 4 \right) + c \\
&= \frac{33}{2} \arcsen \frac{x-2}{5} - \sqrt{21+4x-x^2} \left(\frac{x+6}{2} \right) + c
\end{aligned}$$

$$6.63.- \int \frac{dx}{(x^2 - 2x + 5)^{\frac{3}{2}}}$$

Solución.-



Se tiene: $x-1 = 2 \tan \theta$, $dx = 2 \sec^2 \theta d\theta$, $\sqrt{(x-1)^2 + 2^2} = 2 \sec \theta$

Completando cuadrados se tiene:

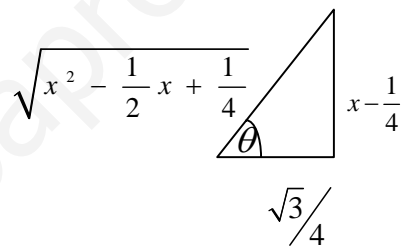
$x^2 - 2x + 5 = (x^2 - 2x + 1) + 5 - 1 = (x^2 - 2x + 1) + 4 = (x-1)^2 + 2^2$, luego:

$$\begin{aligned} \int \frac{dx}{(x^2 - 2x + 5)^{\frac{3}{2}}} &= \int \frac{dx}{\sqrt{[(x-1)^2 + 2^2]^3}} = \int \frac{2 \sec^2 \theta d\theta}{2^3 \sec^3 \theta} = \frac{1}{4} \int \cos \theta d\theta = \frac{1}{4} \sin \theta + c \\ &= \frac{1}{4} \frac{x-1}{\sqrt{x^2 - 2x + 5}} + c \end{aligned}$$

$$6.64.- \int \frac{(2x+1)dx}{\sqrt{(4x^2 - 2x + 1)^3}}$$

Solución.-

Sea: $u = 4x^2 - 2x + 1$, $du = (8x - 2)dx$



Se tiene: $x - \frac{1}{4} = \frac{\sqrt{3}}{4} \tan \theta$, $dx = \frac{\sqrt{3}}{4} \sec^2 \theta d\theta$, $\sqrt{(x - \frac{1}{4})^2 + (\frac{\sqrt{3}}{4})^2} = \frac{\sqrt{3}}{4} \sec \theta$

Completando cuadrados se tiene:

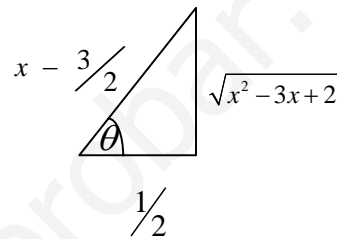
$x^2 - \frac{1}{2}x + \frac{1}{4} = (x^2 - \frac{1}{2}x + \frac{1}{16}) + \frac{1}{4} - \frac{1}{16} = (x - \frac{1}{4})^2 + \frac{3}{16} = (x - \frac{1}{4})^2 + (\frac{\sqrt{3}}{4})^2$, luego:

$$\begin{aligned} \int \frac{(2x+1)dx}{\sqrt{(4x^2 - 2x + 1)^3}} &= \frac{1}{4} \int \frac{(8x+4)dx}{\sqrt{(4x^2 - 2x + 1)^3}} = \frac{1}{4} \int \frac{(8x-2+6)dx}{\sqrt{(4x^2 - 2x + 1)^3}} \\ &= \frac{1}{4} \int \frac{(8x-2)dx}{\sqrt{(4x^2 - 2x + 1)^3}} + \frac{3}{2} \int \frac{dx}{\sqrt{(4x^2 - 2x + 1)^3}} \\ &= \frac{1}{4} \int \frac{du}{(u)^{\frac{3}{2}}} + \frac{3}{2} \int \frac{dx}{\sqrt{4(x^2 - \frac{1}{2}x + \frac{1}{4})^3}} = \frac{1}{4} \int (u)^{-\frac{3}{2}} du + \frac{3}{2} \int \frac{dx}{\sqrt{(x^2 - \frac{1}{2}x + \frac{1}{4})^3}} \\ &= \frac{1}{4} \int (u)^{-\frac{3}{2}} du + \frac{3}{16} \int \frac{dx}{\sqrt{[(x - \frac{1}{4})^2 + (\frac{\sqrt{3}}{4})^2]^3}} = \frac{1}{4} \int (u)^{-\frac{3}{2}} du + \frac{3}{16} \int \frac{\frac{\sqrt{3}}{4} \sec^2 \theta d\theta}{(\frac{\sqrt{3}}{4} \sec \theta)^3} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \int (u)^{-\frac{1}{2}} du + \int \frac{d\theta}{\sec \theta} = \frac{1}{4} \frac{u^{-\frac{1}{2}}}{(-\frac{1}{2})} + \text{sen } \theta + c = -\frac{1}{2u^{\frac{1}{2}}} + \text{sen } \theta + c \\
&= \frac{-1}{2\sqrt{4x^2 - 2x + 1}} + \frac{x - \frac{1}{4}}{\sqrt{x^2 - \frac{1}{2}x + \frac{1}{4}}} + c = \frac{4x - 2}{4\sqrt{x^2 - \frac{1}{2}x + \frac{1}{4}}} + c
\end{aligned}$$

6.65.- $\int \frac{dx}{(x-1)\sqrt{x^2 - 3x + 2}}$

Solución.-



Se tiene: $x - \frac{3}{2} = \frac{1}{2} \sec \theta \Rightarrow x - 1 = \frac{1}{2} (\sec \theta + 1)$, $dx = \frac{1}{2} \sec \theta \tau g \theta d\theta$,

$$\sqrt{(x - \frac{3}{2})^2 + (\frac{1}{2})^2} = \frac{1}{2} \tau g \theta$$

Completando cuadrados se tiene:

$$x^2 - 3x + 2 = (x^2 - 3x + \frac{9}{4}) - \frac{1}{4} = (x - \frac{3}{2})^2 - (\frac{1}{2})^2, \text{ luego:}$$

$$\begin{aligned}
\int \frac{dx}{(x-1)\sqrt{x^2 - 3x + 2}} &= \int \frac{dx}{(x-1)\sqrt{(x - \frac{3}{2})^2 - (\frac{1}{2})^2}} = \int \frac{\frac{1}{2} \sec \theta \tau g \theta d\theta}{\frac{1}{2} (\sec \theta + 1) \frac{1}{2} \tau g \theta} \\
&= \int \frac{\sec \theta d\theta}{\frac{1}{2} (\sec \theta + 1)} = 2 \int \frac{\sec \theta d\theta}{(\sec \theta + 1)} = 2 \int \frac{\sec \theta (\sec \theta - 1) d\theta}{\sec^2 \theta - 1} = 2 \int \frac{\sec^2 \theta d\theta}{\tau g^2 \theta} - 2 \int \frac{\sec \theta d\theta}{\tau g^2 \theta} \\
&= 2 \int \cos^2 \theta d\theta - 2 \int \frac{\text{cosec } \theta d\theta}{\text{sen}^2 \theta} = -2 \text{co } \tau g \theta + 2 \text{cosec } \theta + c \\
&= -2 \frac{1/2}{\sqrt{x^2 - 3x + 2}} + 2 \frac{x - 3/2}{\sqrt{x^2 - 3x + 2}} + c = \frac{2x - 4}{\sqrt{x^2 - 3x + 2}} + c
\end{aligned}$$

6.66.- $\int \frac{xdx}{\sqrt{x^2 - 2x + 5}}$

Solución.-

Se tiene: $x - 1 = 2 \tau g \theta$, $dx = 2 \sec^2 \theta d\theta$, $\sqrt{(x-1)^2 + (2)^2} = 2 \sec \theta$

Completando cuadrados se tiene:

$$x^2 - 2x + 5 = (x^2 - 2x + 1) + 4 = (x - 1)^2 + 2^2, \text{ luego:}$$

$$\int \frac{x dx}{\sqrt{x^2 - 2x + 5}} = \int \frac{x dx}{\sqrt{(x-1)^2 - 2^2}} = \int \frac{(2\tau g\theta + 1) \cancel{\sec^2} \theta d\theta}{2\sec\theta}$$

$$= 2 \int \tau g\theta \sec\theta d\theta + \int \sec\theta d\theta = 2\sec\theta + \ell\eta |\sec\theta + \tau g\theta| + c$$

$$= \sqrt{x^2 - 2x + 5} + \ell\eta \left| \frac{\sqrt{x^2 - 2x + 5} + x - 1}{2} \right| + c$$

6.67.- $\int \frac{(x+1)dx}{\sqrt{2x-x^2}}$

Solución.-

Se tiene: $x-1 = \text{sen } \theta \Rightarrow x+1 = \text{sen } \theta + 2, dx = \cos \theta d\theta, \sqrt{1-(x-1)^2} = \cos \theta$

Completando cuadrados se tiene:

$2x - x^2 = -(x^2 - 2x) = -(x^2 - 2x + 1 - 1) = -(x^2 - 2x + 1) + 1 = 1 - (x-1)^2$, luego:

$$\int \frac{(x+1)dx}{\sqrt{2x-x^2}} = \int \frac{(x+1)dx}{\sqrt{1-(x-1)^2}} = \int \frac{(\text{sen } \theta + 2)\cos\theta d\theta}{\cos\theta} = \int \text{sen } \theta d\theta + 2 \int d\theta$$

$$= -\cos\theta + 2\theta + c = -\sqrt{2x-x^2} + 2\arcsen(x-1) + c$$

6.68.- $\int \frac{(x-1)dx}{\sqrt{x^2-4x+3}}$

Solución.-

Se tiene: $x-2 = \sec\theta \Rightarrow x-1 = \sec\theta + 1, dx = \sec\theta \tau g\theta d\theta, \sqrt{(x-2)^2 - 1} = \tau g\theta$

Completando cuadrados se tiene:

$x^2 - 4x + 3 = x^2 - 4x + 4 - 1 = (x-2)^2 - 1$, luego:

$$\int \frac{(x-1)dx}{\sqrt{x^2-4x+3}} = \int \frac{(x-1)dx}{\sqrt{(x-2)^2 - 1}} = \int \frac{(\sec\theta + 1)\sec\theta \tau g\theta d\theta}{\tau g\theta}$$

$$= \int \sec^2\theta d\theta + \int \sec\theta d\theta = \tau g\theta + \ell\eta |\sec\theta + \tau g\theta| + c$$

$$= \sqrt{x^2 - 4x + 3} + \ell\eta \left| x - 2 + \sqrt{x^2 - 4x + 3} \right| + c$$

6.69.- $\int \frac{dx}{\sqrt{x^2-2x-8}}$

Solución.-

Se tiene: $x-1 = 3\sec\theta, dx = 3\sec\theta \tau g\theta d\theta, \sqrt{(x-1)^2 - 3^2} = 3\tau g\theta$

Completando cuadrados se tiene:

$x^2 - 2x - 8 = x^2 - 2x + 1 - 9 = (x-1)^2 - 3^2$, luego:

$$\int \frac{dx}{\sqrt{x^2-2x-8}} = \int \frac{dx}{\sqrt{(x-1)^2 - 3^2}} = \int \frac{\cancel{\sec\theta} \tau g\theta d\theta}{3\tau g\theta} = \int \sec\theta d\theta = \ell\eta |\sec\theta + \tau g\theta| + c$$

$$= \ell\eta \left| \frac{x-1}{3} + \frac{\sqrt{x^2-2x-8}}{3} \right| + c = \ell\eta \left| x-1 + \sqrt{x^2-2x-8} \right| + c$$

$$6.70.- \int \frac{xdx}{\sqrt{x^2+4x+5}}$$

Solución.-

Se tiene: $x+2 = \tau g\theta$, $dx = \sec^2 \theta d\theta$, $\sqrt{(x+2)^2+1^2} = \sec\theta$

Completando cuadrados se tiene:

$x^2+4x+5 = (x^2+4x+4)+1 = (x+2)^2+1^2$, luego:

$$\int \frac{xdx}{\sqrt{x^2+4x+5}} = \int \frac{xdx}{\sqrt{(x+2)^2+1^2}} = \int \frac{(\tau g\theta-2)\sec^2 \theta d\theta}{\sec\theta} = \int \tau g\theta \sec\theta d\theta - 2 \int \sec\theta d\theta$$

$$= \sec\theta - 2\ell\eta |\sec\theta + \tau g\theta| + c = \sqrt{x^2+4x+5} - 2\ell\eta |\sqrt{x^2+4x+5} + x+2| + c$$

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