

UNIT 7: TRIGONOMETRY.

Trigonometry: Trigonometry (from Greek trigonom “triangle” and metron “measure”) is a branch of mathematics that studies triangles and the relationships between their sides and their angles.

Units of measurement of angles: There are two commonly used units of measurement of angles. The more familiar unit of measurement is **degrees**. A circle is divided into 360 equal degrees, so that a right angle is 90° .

The other common measurement for angles is **radians**.

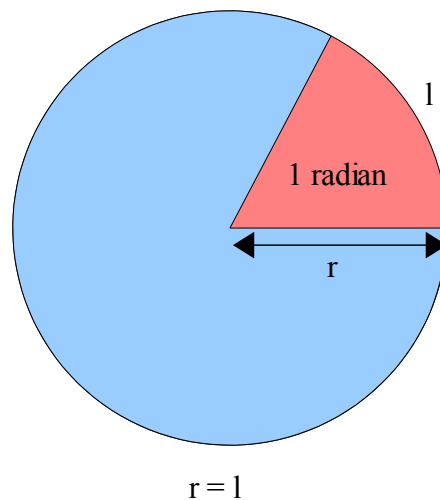
One radian is the angle subtended at the centre of a circle by an arc that is equal in length to the radius of the circle.

Since the length of a circle is $2\pi r$:

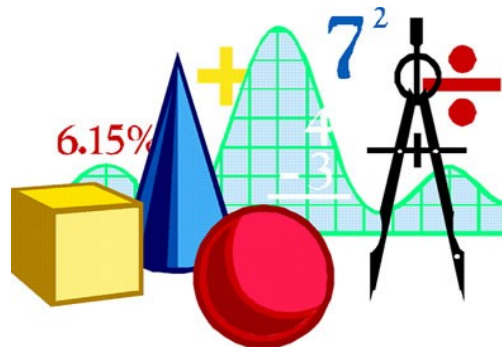
$$360^\circ = 2\pi \text{ radians}$$

or

$$180^\circ = \pi \text{ radians}$$



Your Turn



1. Convert $\frac{\pi}{4}$ radians to degrees.
2. Convert $\frac{2\pi}{3}$ radians to degrees.

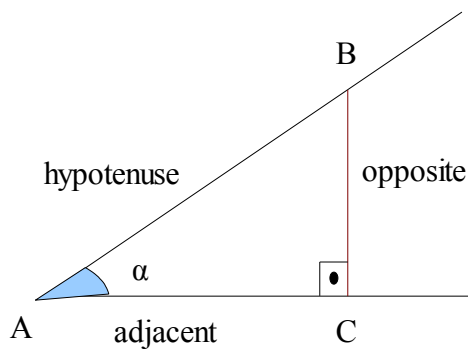
3. Convert 150° to radians.

4. Convert 135° to radians.

5. Complete the following table:

degrees	0°	30°	45°	60°	90°	180°	270°	360°	
radians									1 radian

Trigonometric ratios of an angle: When we draw a perpendicular line to one side of the angle α we get a right-angled triangle. The legs of this triangle are called **opposite** and **adjacent**.



Opposite leg is opposite the angle α , and adjacent leg is adjacent (next) to the angle α .

There are six ways to form ratios of the three sides of this triangle, and each of these ratios has a name:

$$\text{sine of } \alpha = \frac{\textit{opposite leg}}{\textit{hypotenuse}}$$

$$\sin \alpha = \frac{\overline{BC}}{\overline{AB}}$$

$$\text{cosine of } \alpha = \frac{\textit{adjacent leg}}{\textit{hypotenuse}}$$

$$\cos \alpha = \frac{\overline{AC}}{\overline{AB}}$$

$$\text{tangent of } \alpha = \frac{\textit{opposite leg}}{\textit{adjacent leg}}$$

$$\tan \alpha = \frac{\overline{BC}}{\overline{AC}}$$

And their reciprocal (that is, the inverted fractions):

$$\text{cosecant of } \alpha = \frac{\textit{hypotenuse}}{\textit{opposite leg}}$$

$$\csc \alpha = \frac{\overline{AB}}{\overline{BC}} = \frac{1}{\sin \alpha}$$

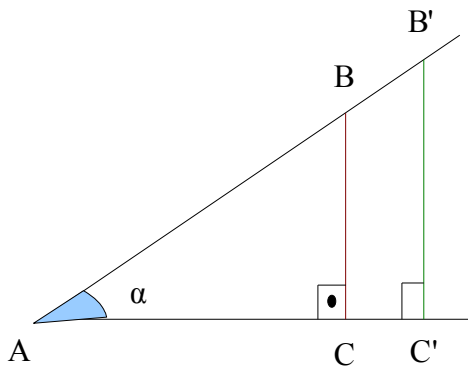
$$\text{secant of } \alpha = \frac{\textit{hypotenuse}}{\textit{adjacent leg}}$$

$$\sec \alpha = \frac{\overline{AB}}{\overline{AC}} = \frac{1}{\cos \alpha}$$

$$\text{cotangent of } \alpha = \frac{\textit{adjacent leg}}{\textit{opposite leg}}$$

$$\cot \alpha = \frac{\overline{AC}}{\overline{BC}} = \frac{1}{\tan \alpha}$$

If we drew other different right-angled triangles for the same angle, what would happen to the trigonometric ratios?



In ABC: $\sin \alpha = \frac{BC}{AB}$

In AB'C' : $\sin \alpha = \frac{B'C'}{AB'}$

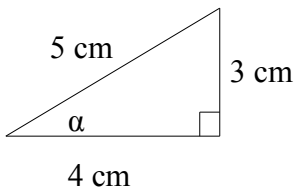
Since the triangles ABC and AB'C' are similar,
 $\frac{BC}{AB} = \frac{B'C'}{AB'}$.

Therefore, the value of the sine of α does not depends on the right-angled triangle that we use.

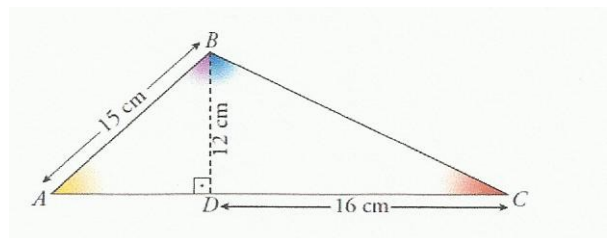
The same thing can be said about the other trigonometric ratios.

Examples:

1. Find the trigonometric ratios of the angle α :



2. Find the sine, cosine and tangent of the angles \hat{A} , \hat{C} , \widehat{ABD} and \widehat{CBD}



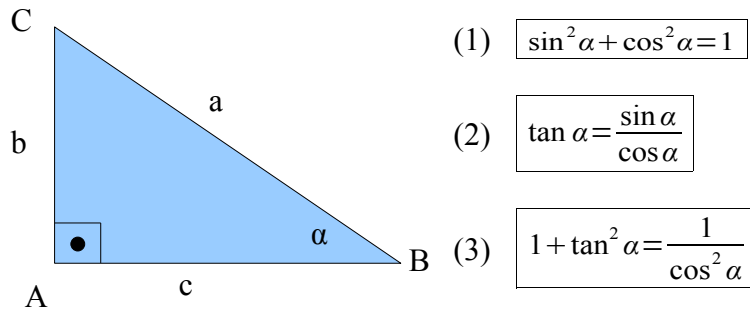
Relations between the trigonometric ratios of an angle:

The trigonometric ratios of an angle are not independent. They have same relations between them.

*Note: We use the following notation to indicate the powers of the trigonometric ratios:

$$(\sin \alpha)^2 = \sin^2 \alpha \quad (\cos \alpha)^2 = \cos^2 \alpha \quad (\tan \alpha)^2 = \tan^2 \alpha$$

The **fundamental relationships** between the trigonometric ratios of an angle are:

**Proof:**

$$(1) \quad \sin^2 \alpha + \cos^2 \alpha = 1$$

$$(2) \quad \tan \alpha = \frac{\sin \alpha}{\cos \alpha}$$

$$(3) \quad 1 + \tan^2 \alpha = \frac{1}{\cos^2 \alpha}$$

Examples:

1. If $\sin \alpha = \frac{4}{5}$, α an acute angle, find the values of $\cos \alpha$ and $\tan \alpha$.

2. If $\cos \alpha = \frac{5}{13}$, α an acute angle, find the $\sin \alpha$ and $\tan \alpha$.

3. If $\tan \alpha = 2$, α an acute angle, find the $\sin \alpha$ and $\cos \alpha$.

4. If $\cos \alpha = \frac{2}{3}$, α an acute angle, find the $\sin \alpha$ and $\tan \alpha$.

5. If $\tan \alpha = \sqrt{5}$, α an acute angle, find the $\sin \alpha$ and $\cos \alpha$.

6. Use the calculator to complete the following table:

α	52°	$34^\circ 23'$	$32^\circ 43' 56''$	$12^\circ 23' 34''$
$\sin \alpha$				
$\cos \alpha$				
$\tan \alpha$				

7. Find the angle α , using your calculator, in each case. Express the angles in degrees, minutes and seconds.

a) $\sin \alpha = 0.58$

b) $\cos \alpha = 0.75$

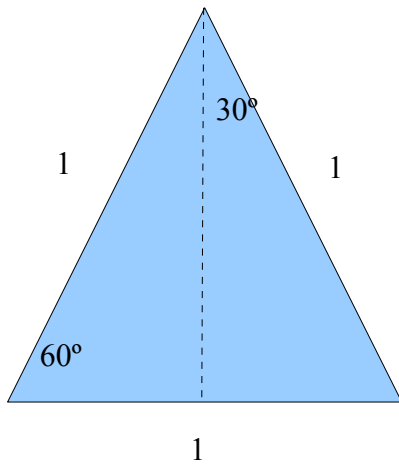
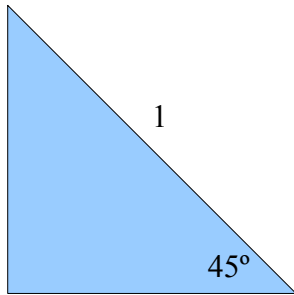
c) $\tan \alpha = 2.5$

d) $\sin \alpha = \frac{\sqrt{5}}{3}$

e) $\cos \alpha = \frac{1}{\sqrt{3}}$

f) $\tan \alpha = 1$

Trigonometric ratios of 30°, 45° and 60°:

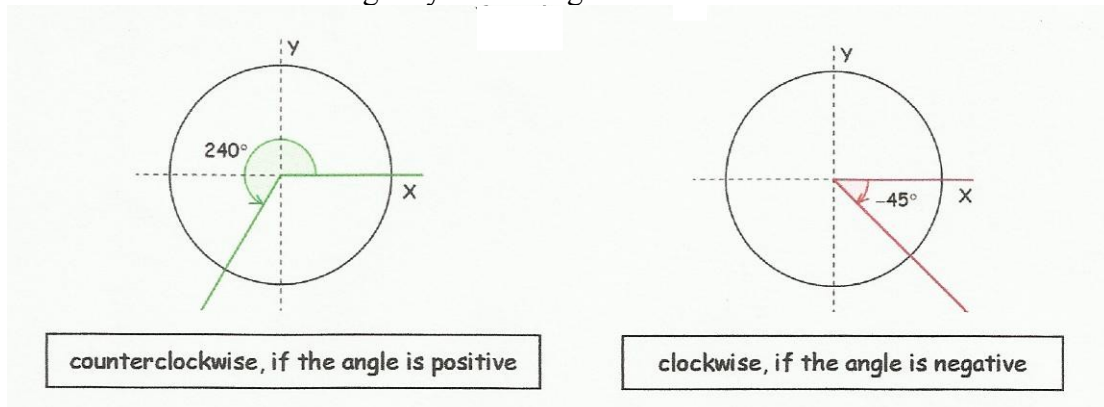


	30°	45°	60°
sin			
cos			
tan			

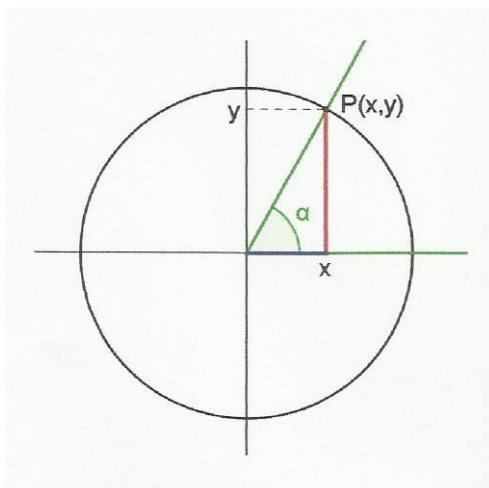
Trigonometric ratios of any angle:

Angles in a circumference: We can represent angles in a circumference which centre is the origin of coordinates:

- Take the origin of coordinates as a vertex of the triangle.
- Use the radius on the positive semi-axis of abscissas as origin of the angle.
- Draw the extreme of the angle by measuring it.



If the radius is one, the circumference is called **goniometric circumference**.



If we represent an angle α in this circumference, we get the point $P(x,y)$. This point is the intersection of the side of the angle that is not on the x-axis with the circumference.

$P(x,y)$ is 1 unit away from the origin. Therefore, the sine of α is the value of the y-coordinate of P, and the cosine of α is the value of the x-coordinate of P.

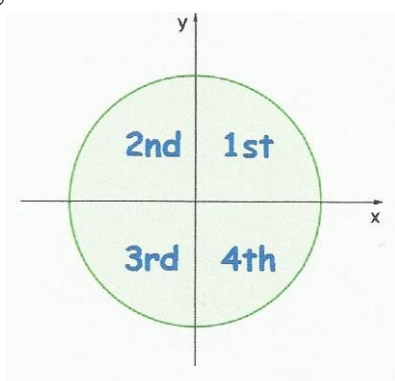
$\sin \alpha = y$

$\cos \alpha = x$

$\tan \alpha = \frac{y}{x}$

Sine, cosine and tangent in the four quadrants:

The coordinates axes divide the plane into four equal parts called **quadrants**. The angles in the goniometric circumference will be each of the four quadrants.

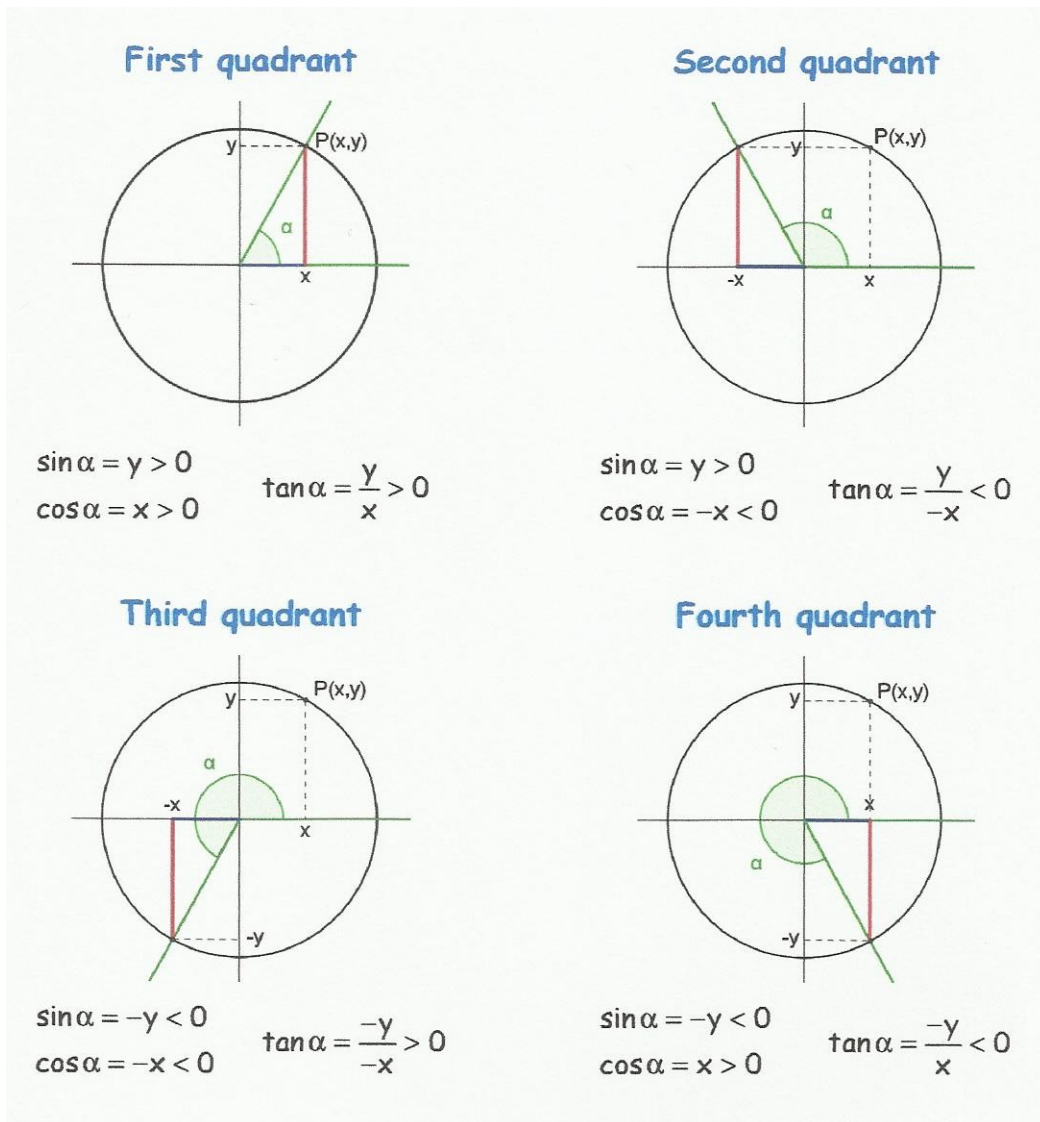


If $0^\circ < \alpha < 90^\circ$, α belongs to the 1st quadrant

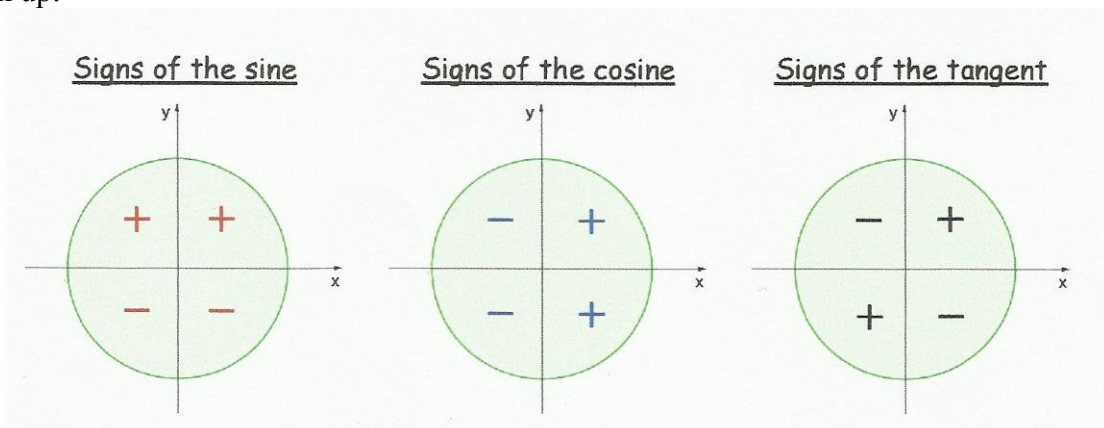
If $90^\circ < \alpha < 180^\circ$, α belongs to the 2nd quadrant

If $180^\circ < \alpha < 270^\circ$, α belongs to the 3rd quadrant

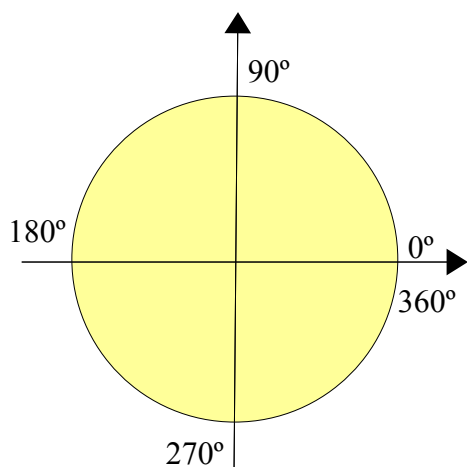
If $270^\circ < \alpha < 360^\circ$, α belongs to the 4th quadrant



To sum up:

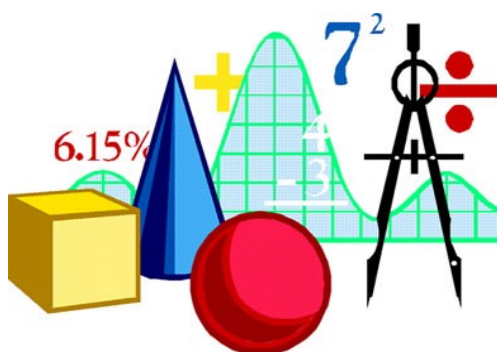


The trigonometric ratios of the angles that coincide on the axis: 0°, 90°, 180°, 270° and 360° are:



Angle	0°	90°	180°	270°	360°
sine					
cosine					
tangent					

Your Turn



1. If $\sin \alpha = \frac{3}{4}$, α an angle of the second quadrant, calculate $\cos \alpha$ and $\tan \alpha$.
2. If $\cos \alpha = \frac{1}{4}$, α an angle of the fourth quadrant, calculate $\sin \alpha$ and $\tan \alpha$.

3. If $\tan \alpha = \frac{1}{2}$, α an angle of the third quadrant, calculate $\sin \alpha$ and $\cos \alpha$.

4. If $\cos \alpha = \frac{-1}{5}$, α an angle of the second quadrant, calculate $\sin \alpha$ and $\tan \alpha$.

5. If $\sin \alpha = \frac{-\sqrt{3}}{3}$, α an angle of the third quadrant, calculate $\cos \alpha$ and $\tan \alpha$.

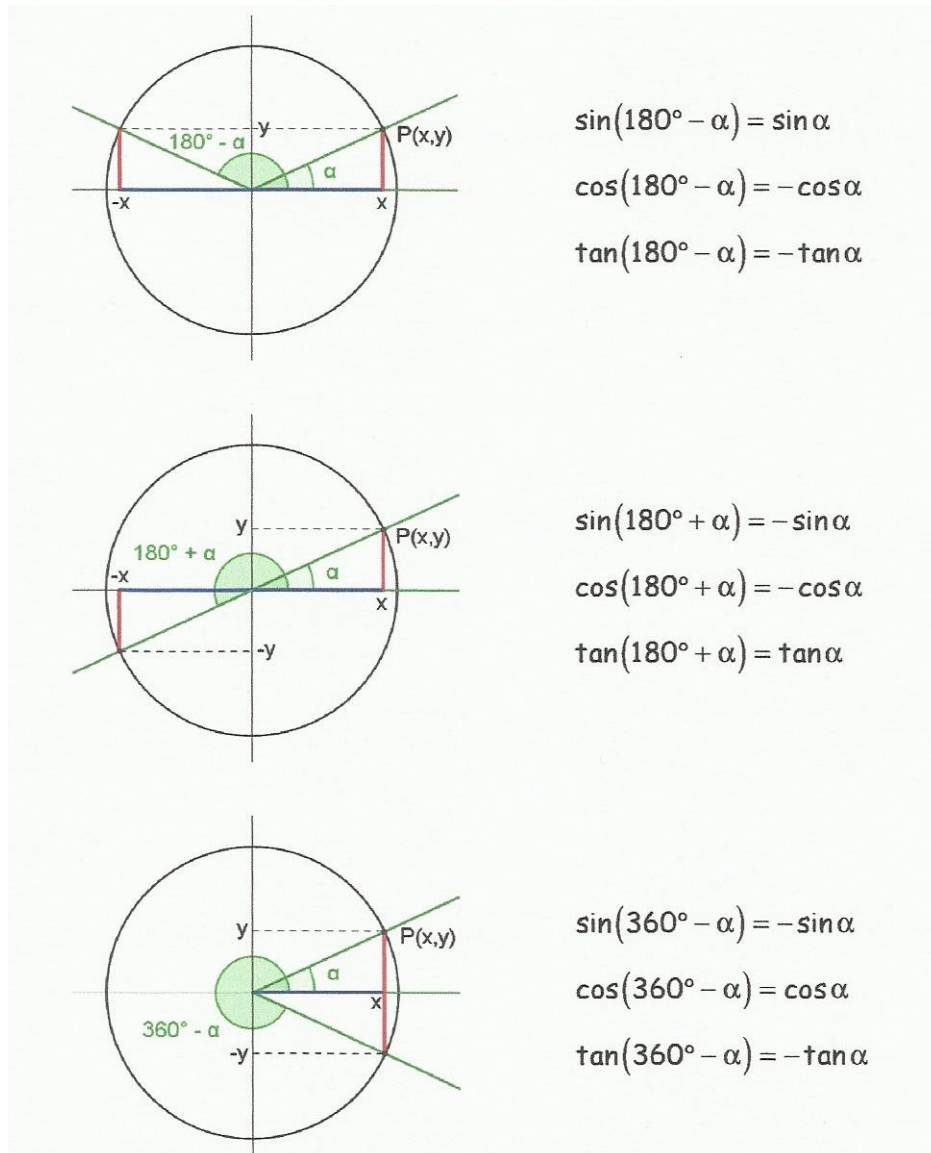
6. If $\tan \alpha = -\sqrt{2}$, α an angle of the fourth quadrant, calculate $\sin \alpha$ and $\cos \alpha$.

How to reduce to the first quadrant the trigonometric ratios of any angle?

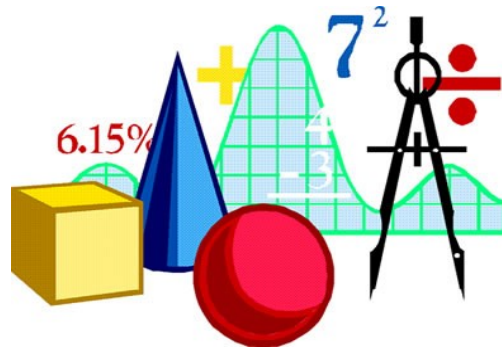
Let α be an angle in the first quadrant.

Then, $180^\circ - \alpha$ is an angle of the second quadrant, $180^\circ + \alpha$ is an angle in the third quadrant and $360^\circ - \alpha$ is an angle in the fourth quadrant.

The trigonometric ratios of $180^\circ - \alpha$, $180^\circ + \alpha$ and $360^\circ - \alpha$ can be expressed in terms of the trigonometric ratios of α .

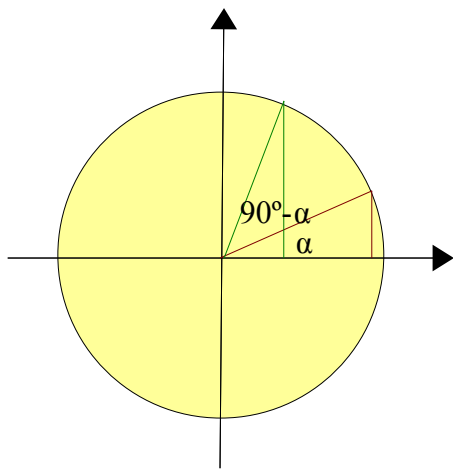


Your Turn



1. If α is an acute angle and $\cos \alpha = \frac{5}{9}$ what are the trigonometric ratios of the angle $180^\circ + \alpha$?
2. Use the trigonometric ratios of 45° to find the trigonometric ratios of 135° , 225° and 315° .
3. Use the trigonometric ratios of 30° to find the trigonometric ratios of 150° , 210° and 330° .
4. Use the trigonometric ratios of 60° to find the trigonometric ratios of 120° , 240° and 300° .

Trigonometric ratios of complementary angles: α and $90^\circ - \alpha$.



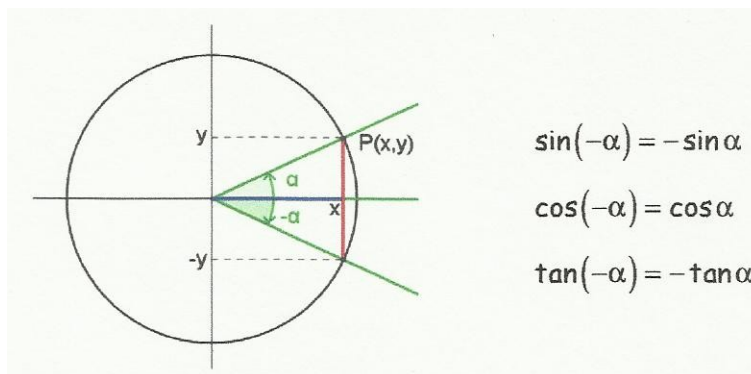
$$\sin (90^\circ - \alpha) = \cos \alpha$$

$$\cos (90^\circ - \alpha) = \sin \alpha$$

$$\tan (90^\circ - \alpha) = \cotan \alpha$$

Example: If $\sin 20^\circ = 0,342$, calculate without using calculator, the trigonometric ratios of the angle of 70° .

Trigonometric ratios of negative angles:

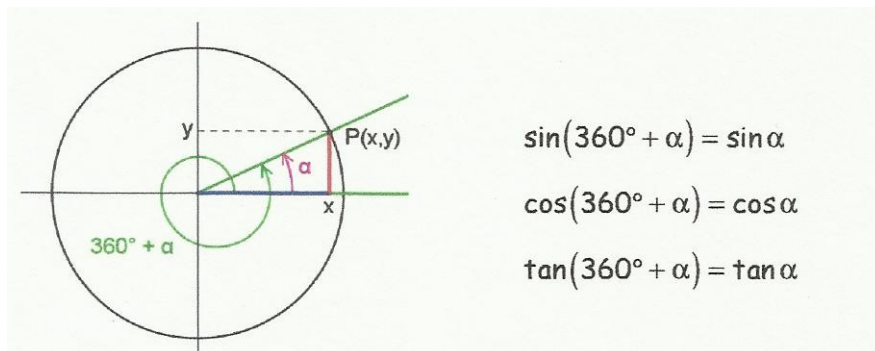


$$\sin(-\alpha) = -\sin \alpha$$

$$\cos(-\alpha) = \cos \alpha$$

$$\tan(-\alpha) = -\tan \alpha$$

Trigonometric ratios of angles greater than 360° :



$$\sin(360^\circ + \alpha) = \sin \alpha$$

$$\cos(360^\circ + \alpha) = \cos \alpha$$

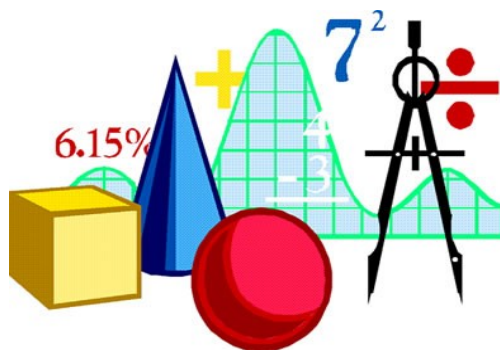
$$\tan(360^\circ + \alpha) = \tan \alpha$$

In general, the trigonometric ratios of $n \cdot 360^\circ + \alpha$, where $n \in \mathbb{Z}$ and $0^\circ \leq \alpha < 360^\circ$, are the same as the trigonometric ratios of α .

Example: $\sin 1500^\circ = \sin (4 \cdot 360^\circ + 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$

$$\begin{array}{r} 1500^\circ \\ 60^\circ \end{array} \quad \begin{array}{l} \underline{360^\circ} \\ 4 \text{ rotations} \end{array}$$

*Your
Turn*



1. Without using calculator, find the values of these trigonometric ratios:

a) $\sin 120^\circ$

b) $\cos 150^\circ$

c) $\tan 120^\circ$

d) $\sin (-60^\circ)$

e) $\cos 1125^\circ$

f) $\tan 930^\circ$

g) $\sin 210^\circ$

h) $\cos (-330^\circ)$

i) $\tan (-315^\circ)$

j) $\sin (-330^\circ)$

k) $\cos (-1305^\circ)$

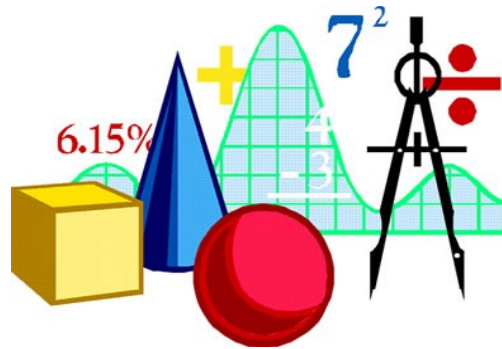
l) $\tan (210^\circ)$

Solving the triangle:

What does “solving” the triangle means? It means that if we are given some facts about a triangle, we can find some or all the rest. For example, if we know two sides of a right-angled triangle, we can find the third side using Pythagoras' Theorem.

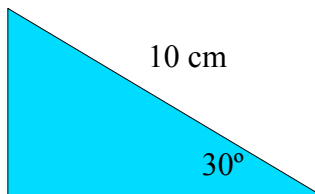
To completely solve a triangle it usually means finding everything about it, all three sides and all three angles.

Your Turn

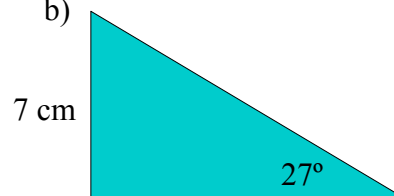


1. Solve the following triangles:

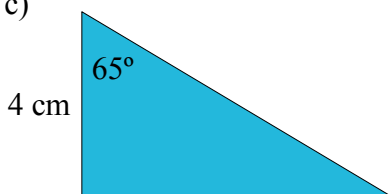
a)



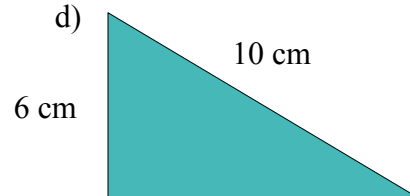
b)

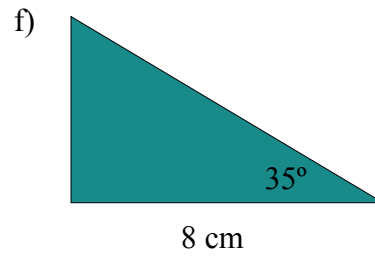
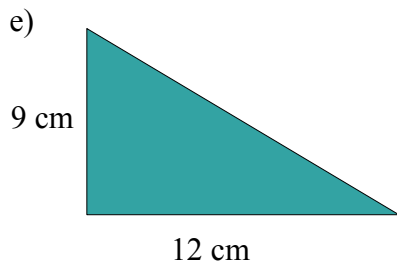


c)

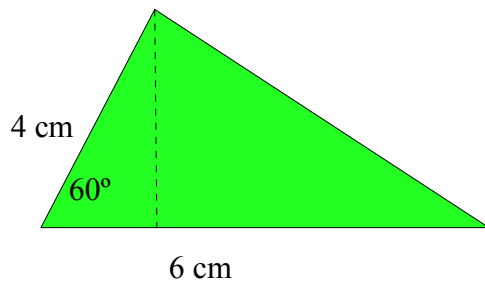


d)



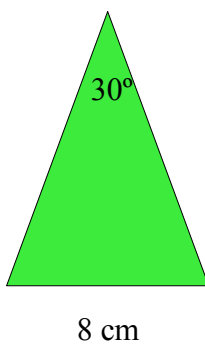


2. Calculate the area of the triangle:

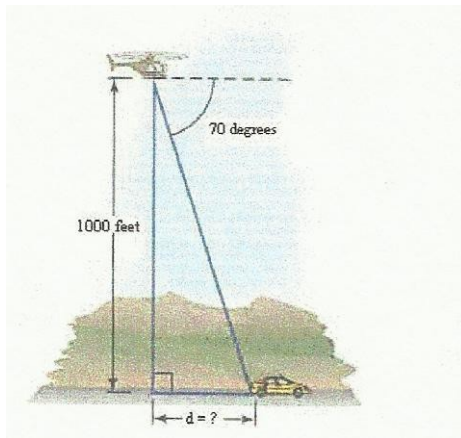


3. Calculate the area of a regular pentagon whose side is 10 cm.

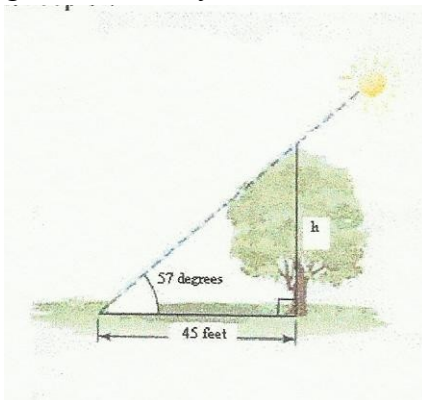
4. Calculate the perimeter and the area of an isosceles triangle whose base is 8 cm and its opposite angle is 30°.



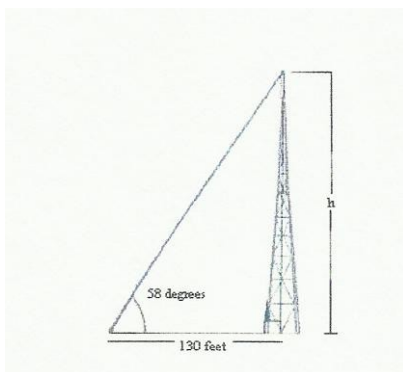
5. The picture below shows a police helicopter chasing a stolen car. The sighted angle of depression is 70 degrees. How far is the helicopter from the car? Find the distance from the stolen car to a point directly below the helicopter?



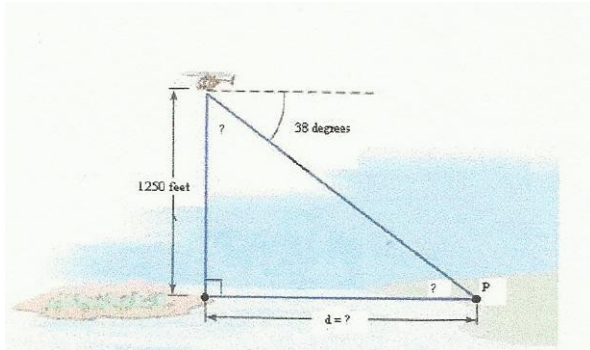
6. At a certain day, the angle of elevation of the sun is 57 degrees. Find the height of the tree if its shadow length is 45 feet. What is the distance from the top of the tree to the point in the ground made by the shadow?



7. The figure below shows a 58 degrees angle of elevation from a point on the ground that is 130 feet from the base of a radio tower to its top. How tall is the radio tower?



8. The helicopter in the picture is hovering 1250 feet above a small island. The angle of depression from the helicopter to the point P on the mainland is 38 degrees. How far is the island from the point P on the mainland? How far is the helicopter from the point P on the mainland? What are the degree measures of the two “?” shown in the picture?



9. The tallest television-transmitting tower in the world is in the state of North Dakota. From a point on level ground one mile from the base of the tower, the angle of elevation is 21 degrees. To the nearest foot, how high is this tower?
10. A balloon is hovering 800 ft. above a lake. The balloon is observed by the crew of a boat as they look upwards at an angle of 20 degrees. Twenty-five seconds later, the crew has to look at an angle of 65 degrees to see the balloon. How faster was the boat travelling?
11. Village A is 15 km away from Village B. From both villages, you can see a balloon in the air. The angle of elevation from village A to the balloon is 30° and the angle of elevation from Village B to the balloon is 20° . How high in the air is the balloon?

12. Maria and Anthony are standing 5 metres apart and are looking up at the top of a tree. Maria is closer to the tree than Anthony. The angle of elevation from where Maria is standing is 45° . The angle of elevation from where Anthony is standing to the top of the tree is 30° . How tall is the tree? How far away from the tree is each person standing?

Keywords:

Trigonometry = **Trigonometría**

angle = **ángulo**

degree = **grado**

radian = **radián**

trigonometric ratios = **razones trigonométricas**

opposite leg = **cateto opuesto**

adjacent leg = **cateto adyacente, cateto contiguo**

hypotenuse = **hipotenusa**

sine = **seno**

cosine = **coseno**

tangent = **tangente**

cosecant = **cosecante**

secant = **secante**

cotangent = **cotangente**

goniometric circumference = **circunferencia goniométrica, circunferencia unidad**

quadrant = **cuadrante**