UNIT 8: FUNCTIONS.

Definition of a function: A function is a relation between two variables such that for every value of the first variable, there is only **one** corresponding value of the second variable. We say that the second variable is a **function** of the first variable.

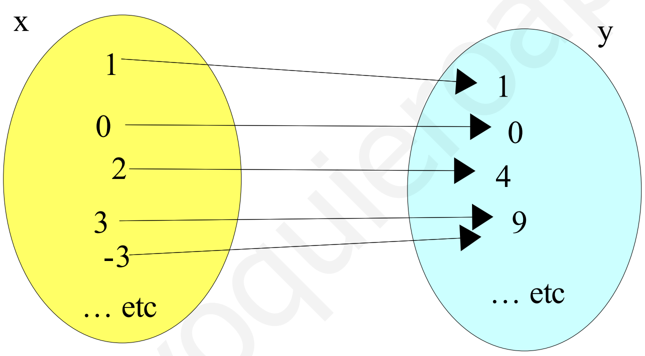
The first variable is the **independent variable** (usually x), and the second variable is the **dependent variable** (usually y).

Examples:

a) You know the formula for the area of a circle $A=\pi \cdot r^2$. This is a function as each value of the independent variable *r* gives you one value of the dependent variable *A*.

b) The force *F* required to accelerate an object of mass 5 kg by an acceleration *a* is given by: $F=5a$. Here *F* is a function of the acceleration, *a*. The dependent variable is *F* and the independent variable is *a*.

c) In the equation $y=x^2$, *y* is a function of *x*, since for each value of *x*, there is only one value of *y*.



We normally write functions as **f(x)**, and read this is a function of x.

For example, the function $y=x^2-3x+4$ is also written as $f(x)=x^2-3x+4$ (*y* and *f(x)* are the same).

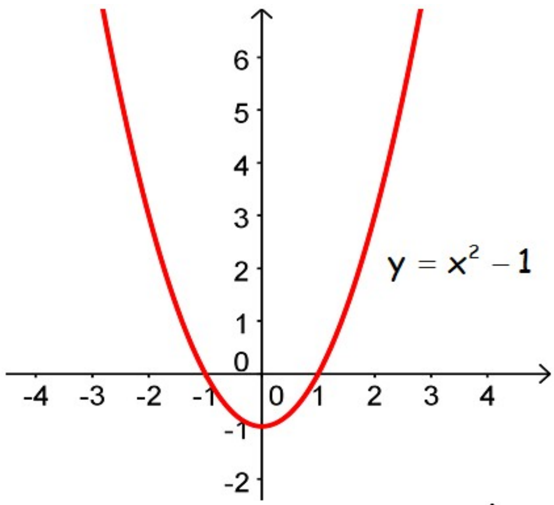
The value of the function f(x) when x=a is f(a).

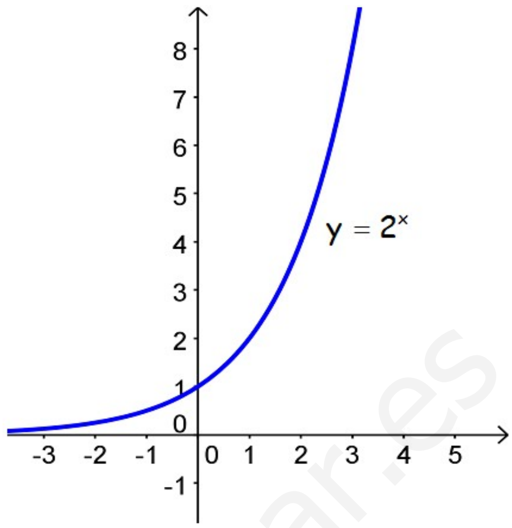
If $f(x)=x^2-3x+4$, then $f(5)=5^2-3\cdot 5+4=25-15+4=14$, and we say 14 is the **image** of 5.

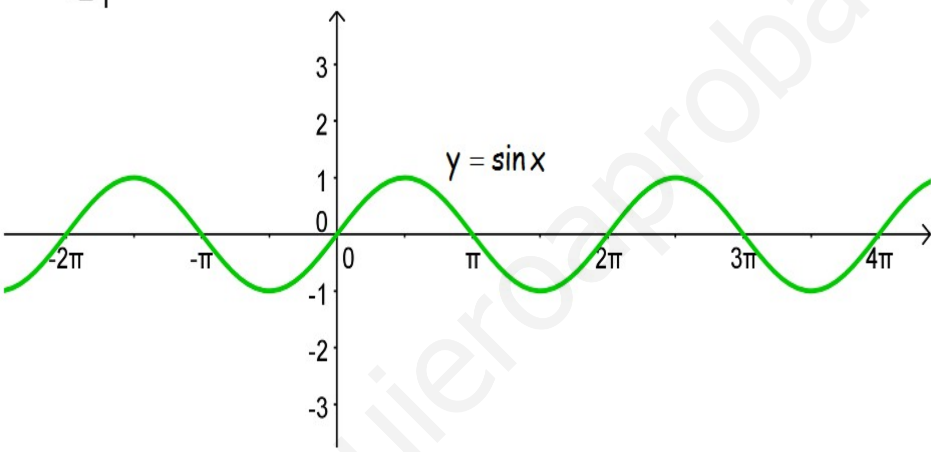
A good way of presenting a function is by **graphical representation**. Graphs give us a visual information about the function.

The values of the independent variable (the x-values) are placed on the horizontal axis (x-axis) and the dependent variable (the y-values) are placed on the vertical axis (y-axis).

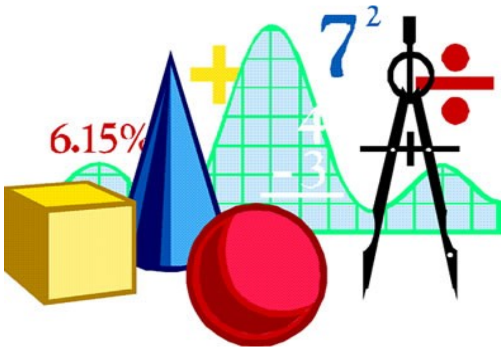
Examples:







Your Turn



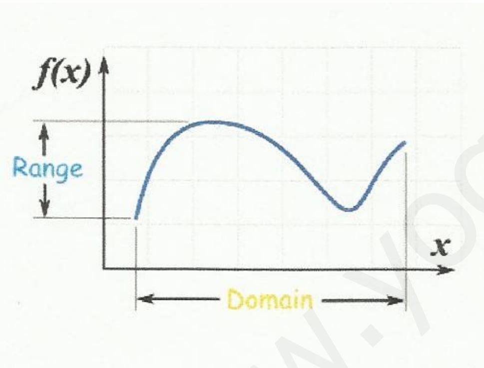
1. The melting point of the ice is 0ºC (or 32ºF) and the boiling point is 100ºC (or 212ºF).

a) Write a linear function that converts any temperature from Celsius degrees (ºC) to Fahrenheit degrees (ºF).

b) Draw the graph of this function.

2. You have a square cardboard. The side of this square is 8 dm. You cut four equal squares from the corners that will allow you to fold up the edges to make a box. The side of this squares is x dm. Express the volume of this box as a function of x.

Domain and Range:

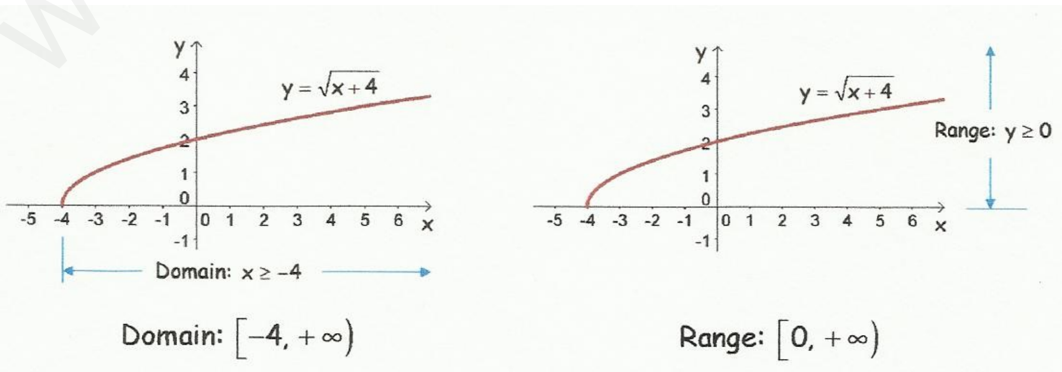


The **domain** of a function is the complete set of possible values of the independent variable of the function.

The **range** (or image) of a function is the complete set of all possible resulting values of the dependent variable of the function, after we have substituted the values in the domain.

Domain of f = Dom f   Range of f = Im f

Example: Find the domain and the range of the function $f(x)=\sqrt{x+4}$.



Domain of some important types of function:

- Polynomial Functions: The domain of all polynomials is $\mathbb{R}$.

  Example: $f(x)=x^2-5x+6 \Rightarrow Dom\ f=\mathbb{R}$

- Rational Functions: We don´t consider the zeroes of the denominator.

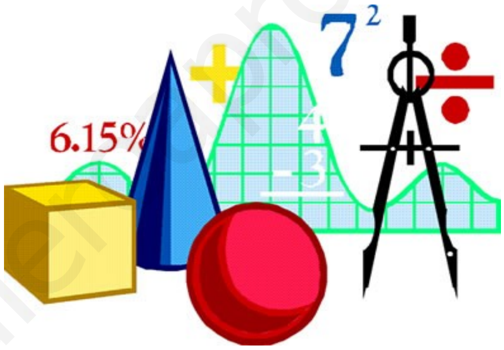
  Example: $f(x)=\dfrac{x+3}{x^2-1} \Rightarrow Dom\ f=\mathbb{R}-\{1,-1\}$

- Irrational Functions: n even $f(x)=\sqrt[n]{x} \Rightarrow Dom\ f=[0,+\infty)$.
  n odd $f(x)=\sqrt[n]{x} \Rightarrow Dom\ f=\mathbb{R}$

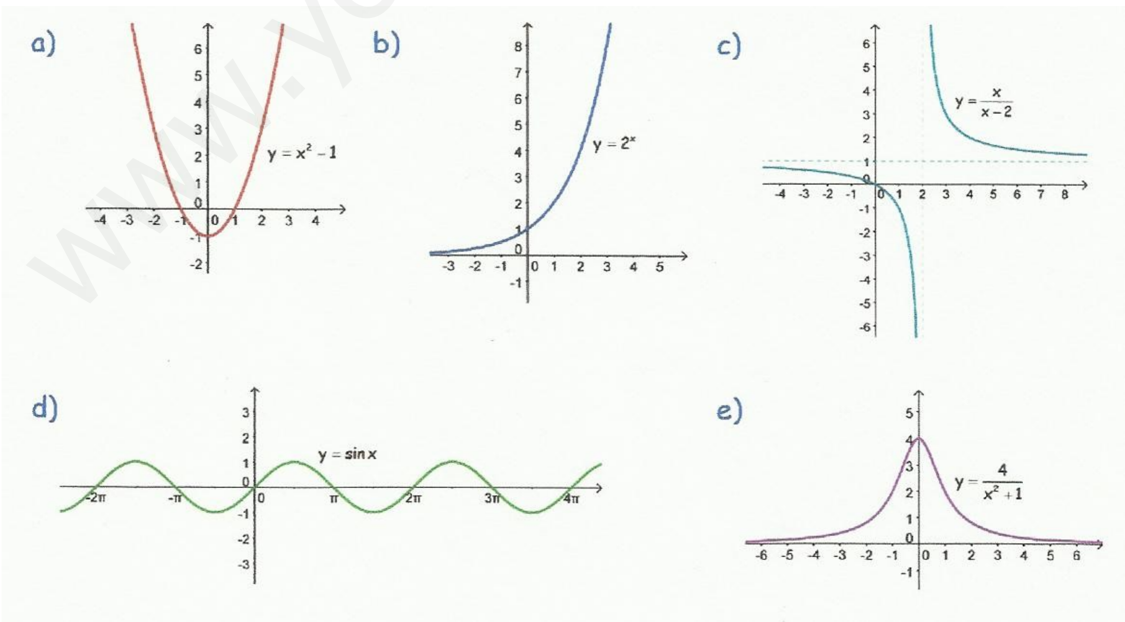
  Examples: $f(x)=\sqrt{x+1}$ $\quad x+1\geqslant 0 \Rightarrow x\geqslant -1 \Rightarrow Dom\ f=[-1,+\infty)$

  $f(x)=\sqrt[3]{x^2-4} \Rightarrow Dom\ f=\mathbb{R}$

Your Turn



1. Find the domain and the range of the following functions:



2. Imagine you have a rope that is 80 cm long. If you join both ends, you can make countless rectangles. Let "x" be one side of this rectangles.

a) Express the area of these rectangles as a function of x.

b) Draw the graph of this function and find its domain and its range.

3. Find the domain of the following functions:

a) $f(x)=\frac{1}{x-4}$

b) $f(x)=\frac{x-1}{x^2+x-6}$

c) $f(x)=x^3-1$

d) $f(x)=\frac{2x}{x^2-3x}$

e) $f(x)=x^4-2x^3+5x-6$

f) $f(x)=\sqrt{2x-6}$

g) $f(x)=\frac{x-4}{x-3}$

h) $f(x)=\sqrt[3]{2x-4}$

i) $f(x)=\sqrt{3x^2+6x-9}$

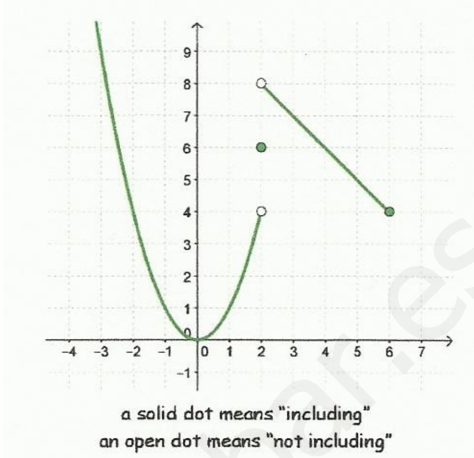
j) $f(x)=\frac{1}{\sqrt{x-7}}$

Piecewise functions:

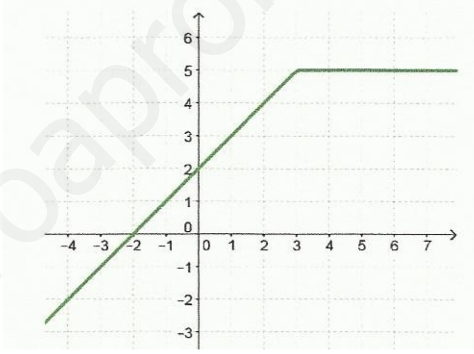
A piecewise function is a function f(x) defined piecewise, that is f(x) is given by different expressions on various intervals.

Examples:

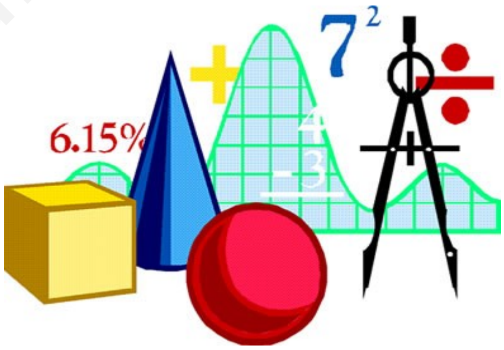
a) $f(x)=\begin{cases} x^2 \ , & if \quad x<2 \\ 6 \ , & if \quad x=2 \\ 10-x \ , & if \quad 2<x\leqslant 6 \end{cases}$



b) $f(x)=\begin{cases} x+2 \ , & if \quad x<3 \\ 5 \ , & if \quad x\geqslant 3 \end{cases}$



Your Turn



1. Graph the following piecewise functions, and write its domain and range:

a) $f(x)=\begin{cases} 2x \ , & if \quad x\leqslant 1 \\ -x+5 \ , & if \quad x>2 \end{cases}$

b) $f(x)=\begin{cases} 2 \ , & if \quad x<-1 \\ x \ , & if \quad -1\leqslant x<2 \\ x-2 \ , & if \quad x\geqslant 3 \end{cases}$

c) $f(x)=\begin{cases} x \ , & if \quad x<0 \\ 1 \ , & if \quad 0\leqslant x\leqslant 2 \\ 2 \ , & if \quad x>2 \end{cases}$

d) $f(x)=\begin{cases} x+3 \ , & if \quad x\leqslant -2 \\ 3 \ , & if \quad -2<x\leqslant 1 \\ -x+2 \ , & if \quad x>1 \end{cases}$

e) $f(x)=\begin{cases} 1 \ , & if \quad x<0 \\ x \ , & if \quad 1<x\leqslant 3 \\ 3 \ , & if \quad x>3 \end{cases}$

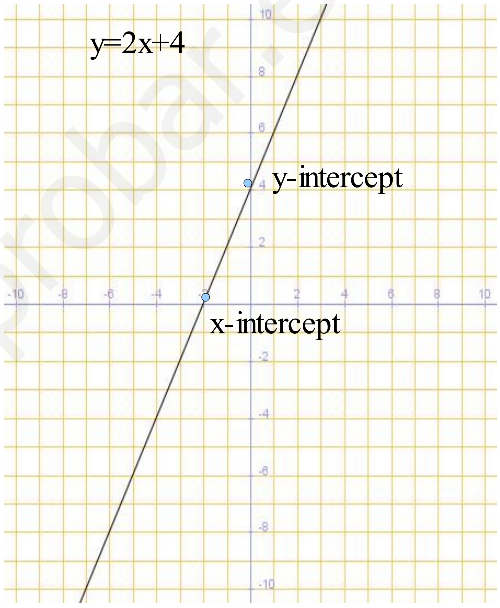
Characteristics of the functions:

Axes Intercepts:

Given the graph of a function:

- An x-intercept is a point where the graph of the function meets the x-axis. x-intercepts are found letting y be 0 in the algebraic expression of the function.
- An y-intercept is the point where the graph of the function meets the y-axis. y-intercepts are found letting x be 0 in the algebraic expression of the function.

Example: Function y=2x+4



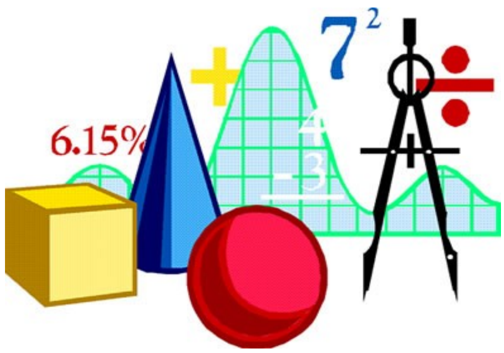
- The graph of this function intercepts the x-axis on the point (-2,0) because:

$$y=0 \Rightarrow 2x+4=0 \Rightarrow 2x=-4 \Rightarrow x=\frac{-4}{2}=-2 \Rightarrow x=-2$$

- The graph of this function intercepts the y-axis on the point (0,4) because:

$$x=0 \Rightarrow y=2 \cdot 0+4=4 \Rightarrow y=4$$

Your Turn

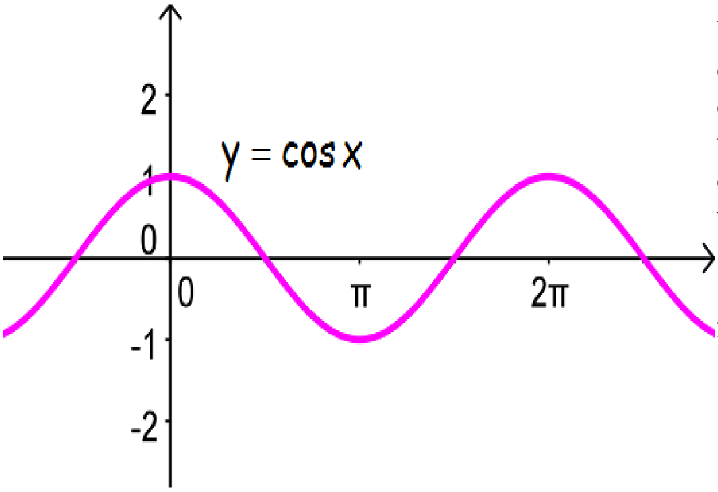


1. Find the axes intercepts of the following functions:

a) $f(x)=x^2-4x+3$ b) $f(x)=2x-6$ c) $f(x)=2x^2-8$

Continuous and discontinuous functions:

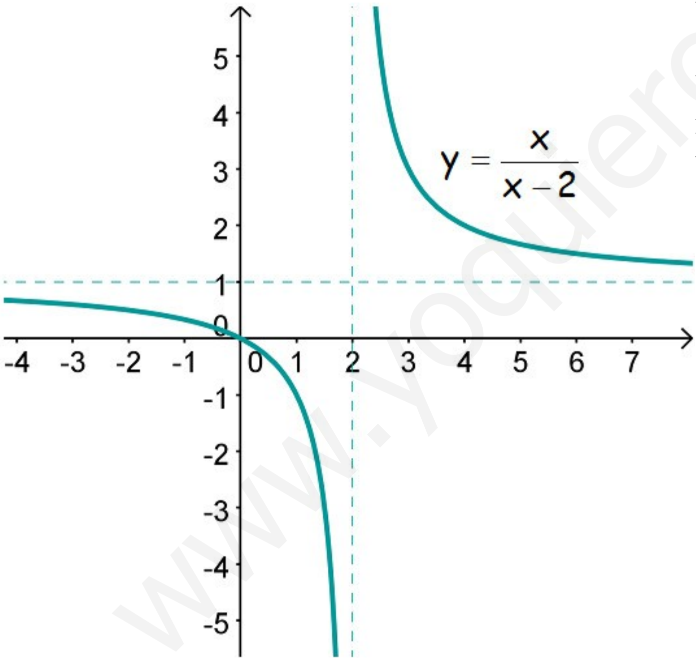
Consider the graph of $y=\cos x$:



We can see that there are no "gaps" in the curve. Any value of x will give us a corresponding value of y. We could continue the graph in the negative and positive directions, and we will never need to take the pencil of the paper.

Such functions are called **continuous functions**.

Now, consider the function $y=\frac{x}{x-2}$.



We can see the curve is **discontinuous** at x=2.

We observe that a small change in x near to x=2, gives a very large change in the value of the function.

Examples: Graph the following functions, and study their continuity:

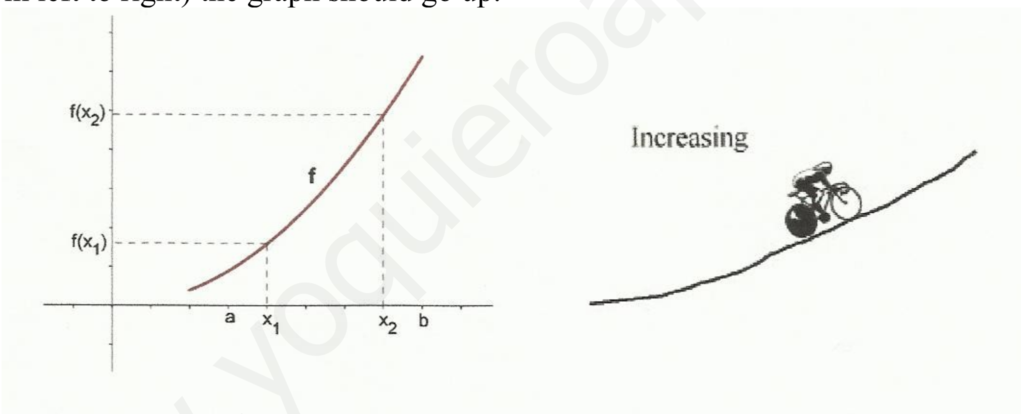
a) $f(x)=\begin{cases} x+2 \ , & if \quad x<1 \\ 2 \ , & if \quad 1\leqslant x<3 \\ x-1 \ , & if \quad x>3 \end{cases}$

b) $f(x)=\begin{cases} x \ , & if \ \ x \leqslant 1 \\ 1 \ , & if \ \ 1 < x \leqslant 2 \\ x-1 \ , & if \ \ x \geqslant 3 \end{cases}$

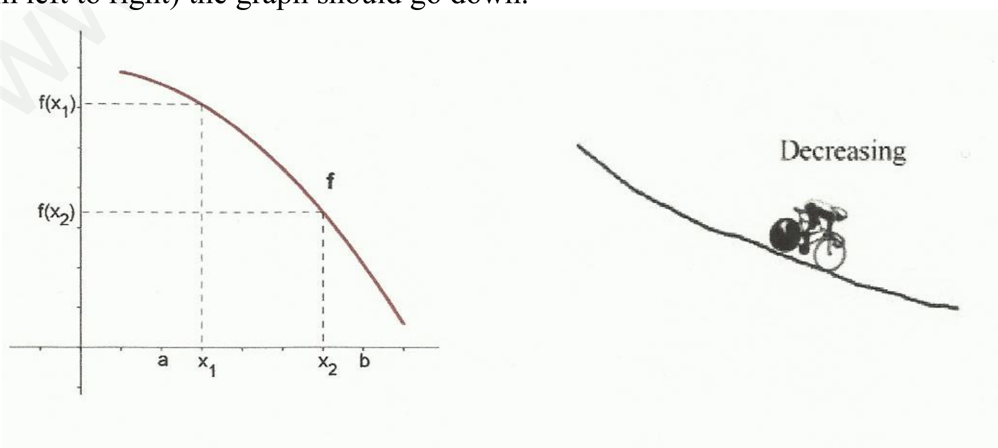
c) $f(x)=\begin{cases} x-1 \ , & if \ \ x \leqslant 0 \\ 4 \ , & if \ \ 0 < x < 3 \\ x \ , & if \ \ x \geqslant 3 \end{cases}$

Increasing and decreasing functions:

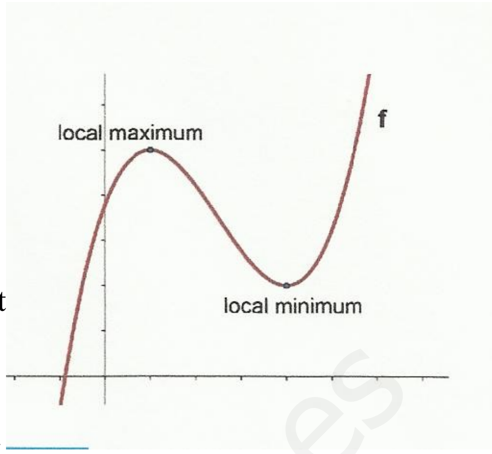
A function f is **increasing** on an interval (a,b) if for any x_1 and x_2 in the interval such that $x_1 < x_2$ then $f(x_1) < f(x_2)$. Another way to look at this is: as you trace the graph from a to b (that is from left to right) the graph should go up.



A function f is **decreasing** on an interval (a,b) if for any x_1 and x_2 in the interval such that $x_1 < x_2$ then $f(x_1) > f(x_2)$. Another way to look at this is: as you trace the graph from a to b (that is from left to right) the graph should go down.



Maxima and minima:

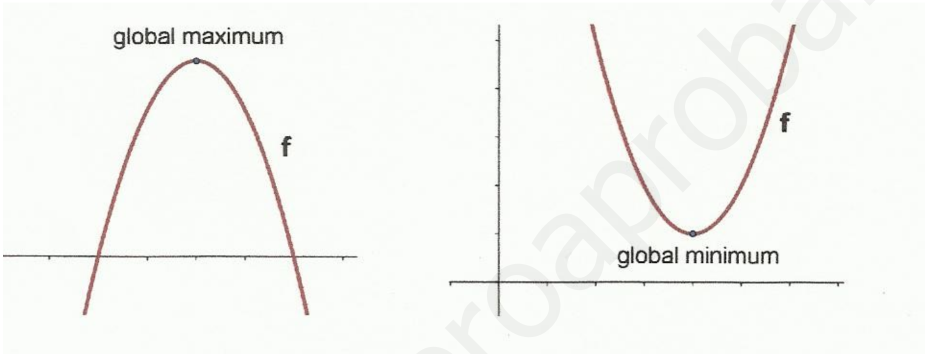


A function f has a **relative** (or **local**) **maximum** at a point if it changes from in increasing to decreasing at this point.

A function f has a **relative** (or **local**) **minimum** at a point if it changes from in decreasing to increasing at this point.

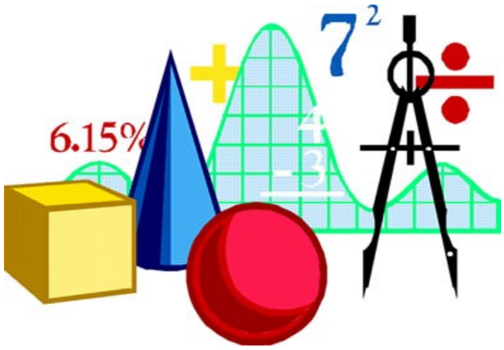
A function f has an **absolute** (or **global**) **maximum** at a point if its ordinate (y-coordinate) is the largest value that the function takes on the domain that we are working on.

A function f has an **absolute** (or **global**) **minimum** at a point if its ordinate (y-coordinate) is the smallest value that the function takes on the domain that we are working on.

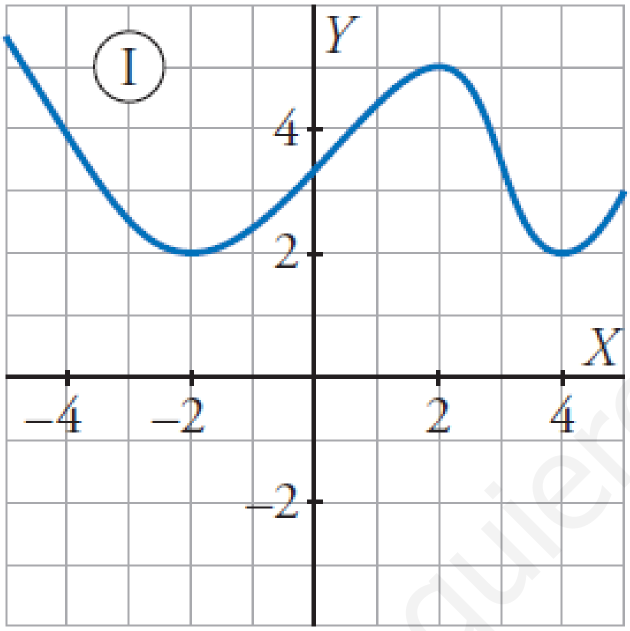


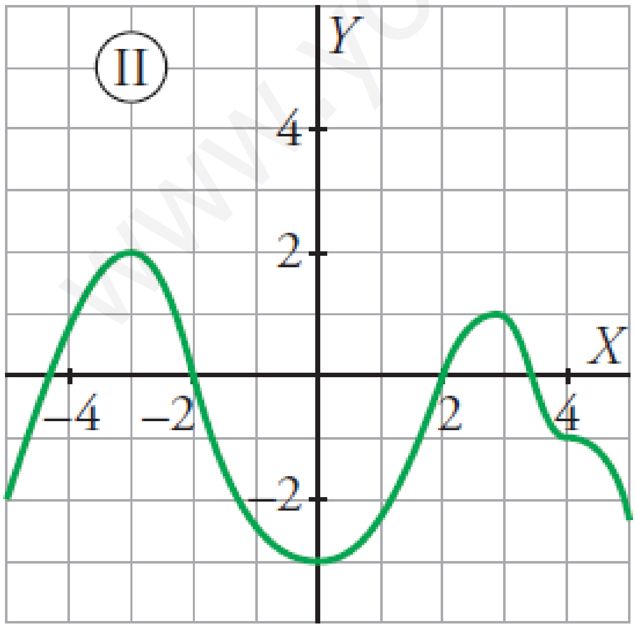
Example: The following graphs shows the present people (in thousands) in shopping centre during a day.

Your Turn

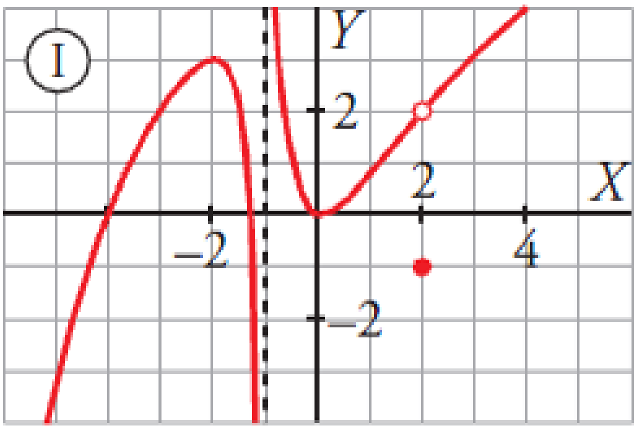


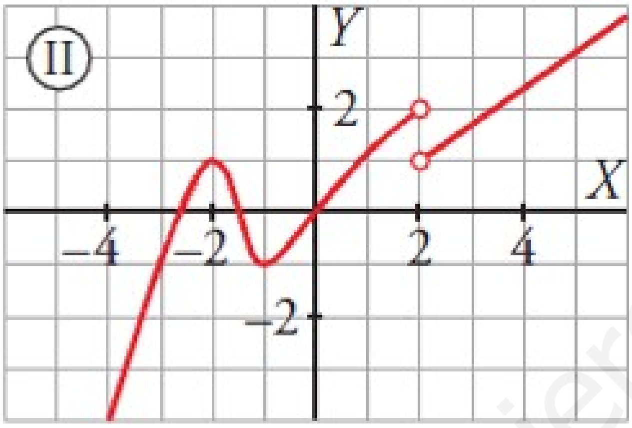
1. Determine the intervals of increasing and decreasing of these functions as well as its maxima and minima.

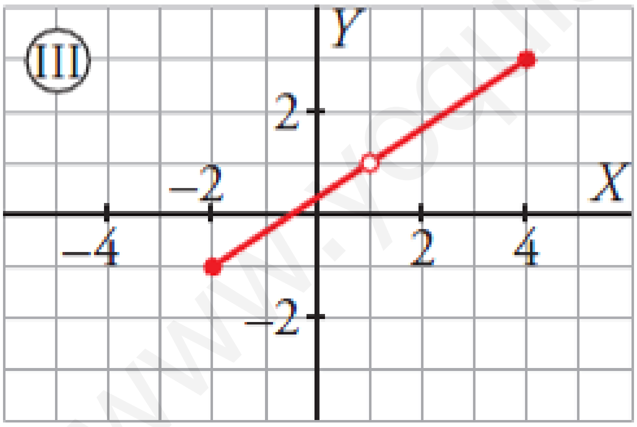


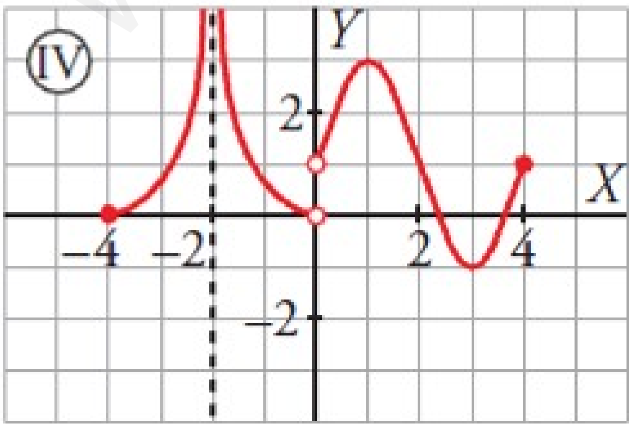


2. Look at these graphs and study the following characteristics: domain, range, continuity, increasing and decreasing intervals, maxima and minima.









3. Draw the graph of the function with the following characteristics:

a) Dom f = $(-\infty, -2] \cup [2,+\infty)$; Im f = $(-\infty, 2]$; relative maxima at the points (-3,2) and (3,2).

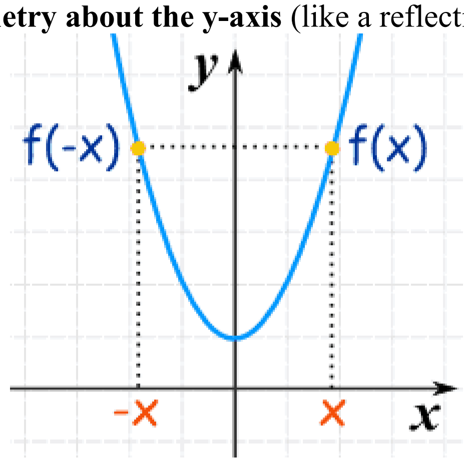
b) Dom g = $\mathbb{R}$; Im g = $(-3,2]$; relative minimum at the point (-2,-1) and relative maximum at the point (0,1).

c) Dom h = $(-\infty, 0)$; Im h = $(1, +\infty)$; increasing in all its domain.

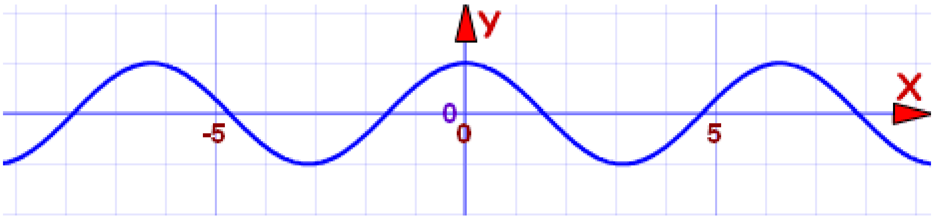
Symmetries. Even and odd functions:

A function is **even** when: f(-x)=f(x) for all x.

In other words, there is a **symmetry about the y-axis** (like a reflection):

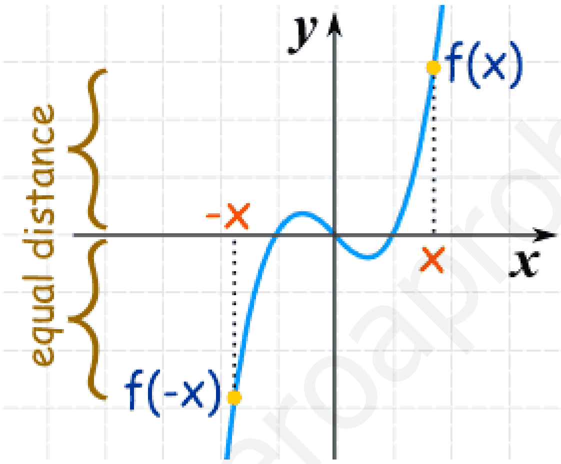


They got called even functions because the functions x^2 , x^4 , x^6 , etc behave like that, but there are other functions that behave like that too, such as cos x:



A function is **odd** when: f(-x)=-f(x) for all x.

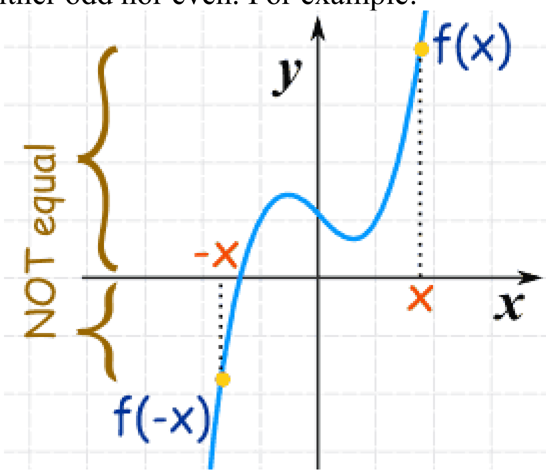
And we get **origin symmetry**:



They got called even functions because the functions x , x^3 , x^5 , etc behave like that, but there are other functions that behave like that too, such as sin x:

Don't be misled by the names odd and even … they are just names … and a function **does not have to be even or odd**.

In fact most functions are neither odd nor even. For example:



Examples: Study if the following functions are odd or even:

a) $f(x)=x^4+x^2$

b) $g(x)=x^3-x$

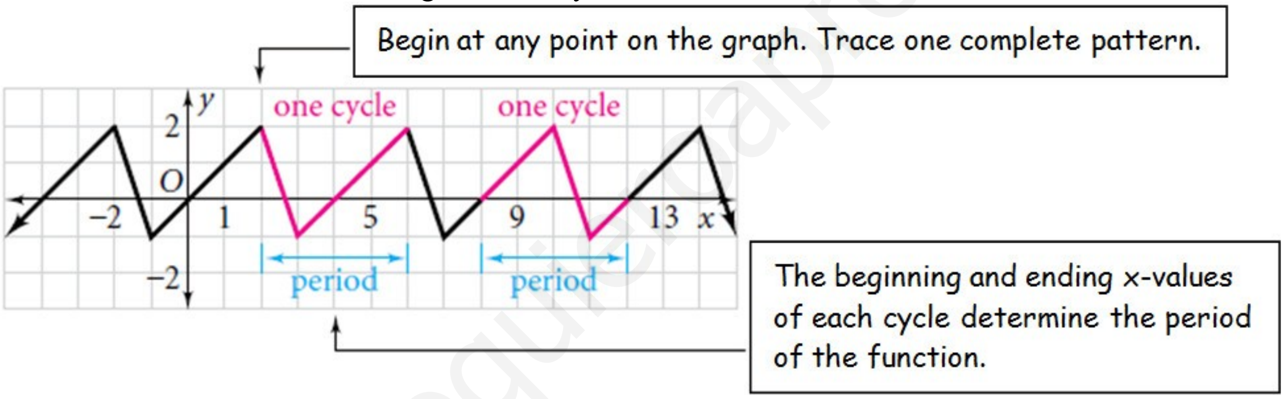
c) $h(x)=x^5+x^2$

d) $i(x)=\dfrac{2}{x^4}$

e) $j(x)=\dfrac{x^2-4}{x^2+4}$

f) $k(x)=\dfrac{x}{x^2-1}$

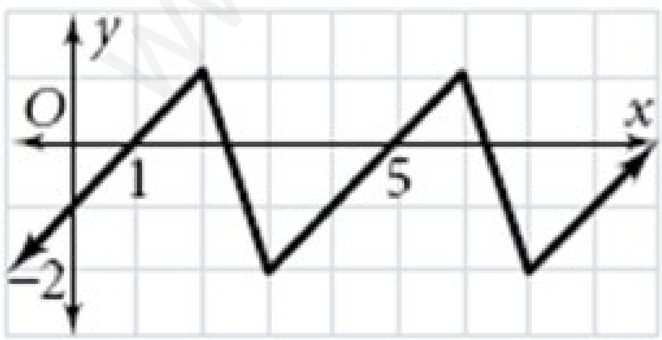
Periodic Functions: A **periodic function** repeat a pattern of y-values at regular intervals. One complete pattern is a **cycle**. A cycle may begin at any point on the graph of the function. The **period** of a function is the horizontal length of one cycle.

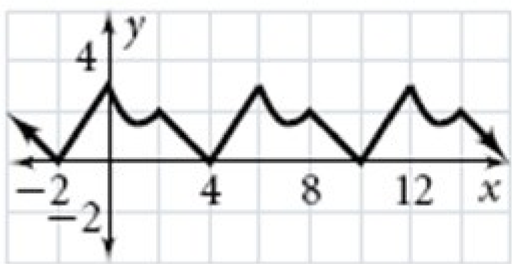


If f is a periodic function whose period is T, then f(x+T)=f(x) for all values of x.

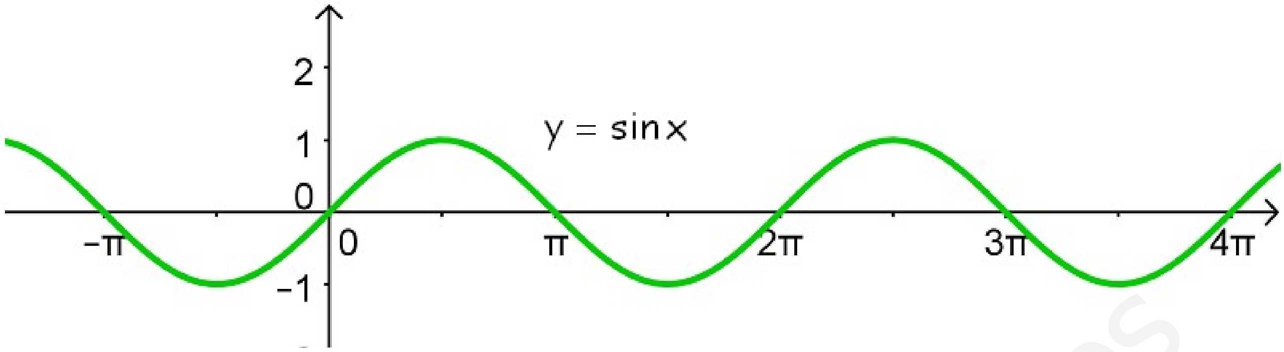
Examples:

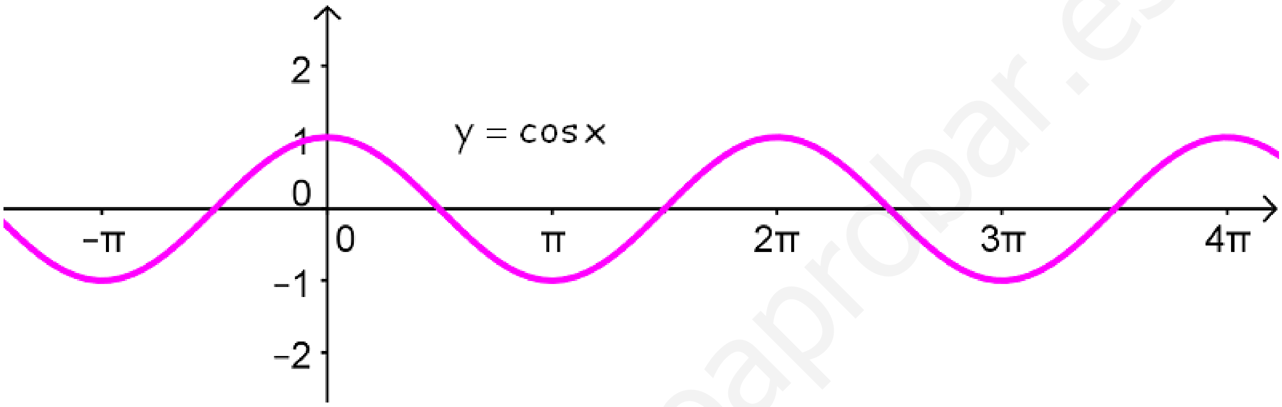
1. Find the period of each function:

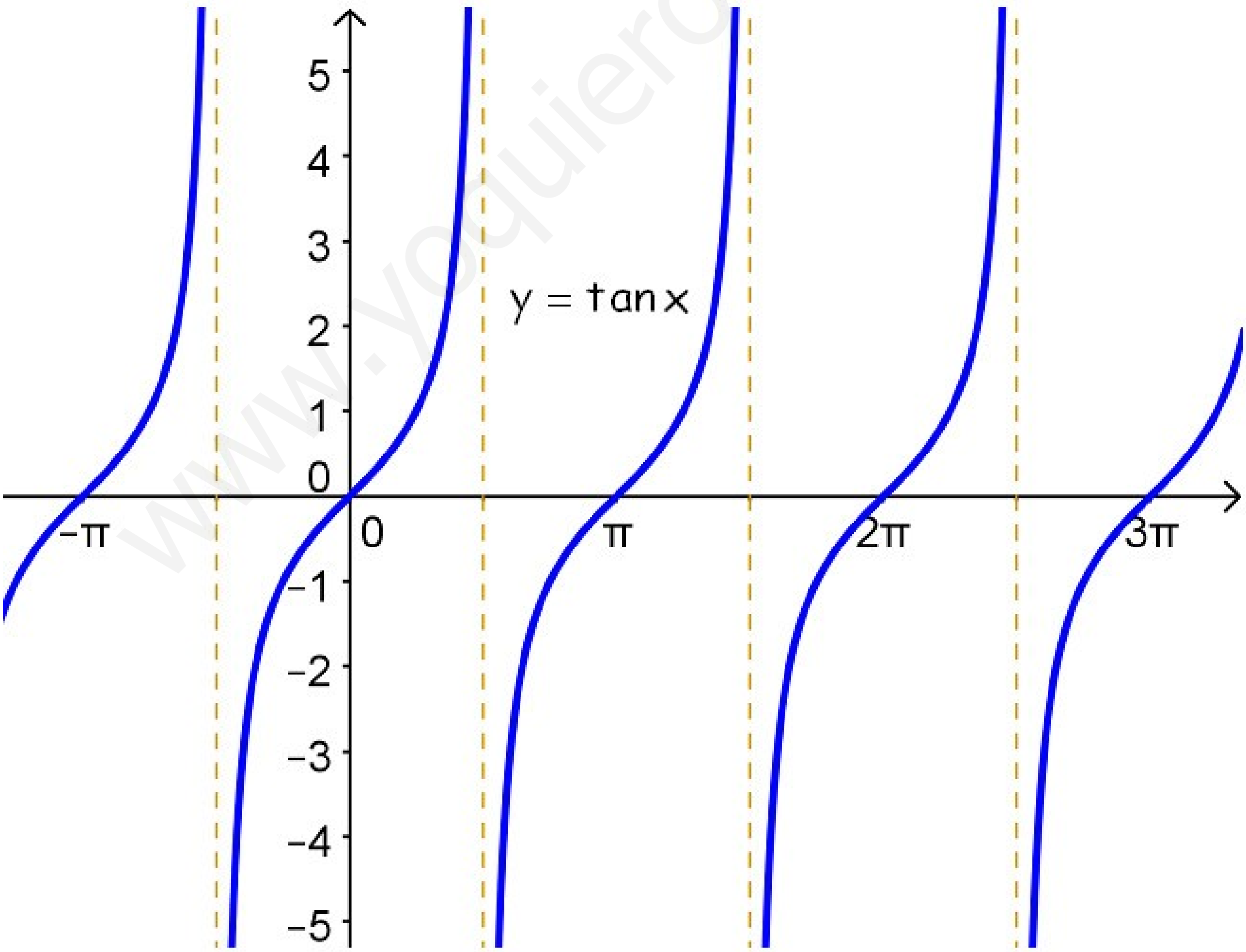




2. The trigonometric functions y=sin x, y=cos x and y=tan x are periodic functions. Look at their graphs and determine their periods.







Keywords:

function=función
independent variable=variable independiente
dependent variable= variable dependiente
coordinate=coordenada
image = imagen
graph = gráfica
domain=dominio
range=recorrido
piecewise functons=funciones definidas a trozos
axes intercepts=puntos de corte (de intersección) con los ejes
continuous functions= funciones continuas
discontinuous functions=funciones discontinuas
continuity=continuidad
increasing function=función creciente
decreasing function=función decreciente
relative maximum/minimum=máximo/mínimo relativo
absolute maximum/minimum=máximo/mínimo relativo
symmetry=simetría
even function=función par o simétrica respecto del eje de ordenadas
odd function=función impar o simétrica respecto el origen.
Periodic function=función periódica
period=periodo